ROBECO

Derivatives for Pension Funds

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April 17, 2007

Abstract

Pension Funds has substantial equity risks and interest rate risk. This study aims at improving the risk return profile of pension fund by incorporating derivatives into the current portfolio. The analysis was carried out using an Asset and Liability (ALM) Model comprising a Vector-Auto Regressive Model for economic scenario generation. Based on historical patterns several derivatives strategies are investigated and compared, followed by tailor-fashion implementation, evaluation and selection. This project at Robeco is twofold; firstly analyze optimal portfolio strategies with derivatives under the new Dutch pension rules FTK, which can potentially be used as input for making strategic asset allocation decisions; secondly investigate the long run effect of including derivatives in the portfolio.

Key Words: Asset & Liability Management (ALM), Pension Funds, Equity Risks, Interest Rate Risks, Equity Options, Interest Rate Swaps, Interest Rate Swaptions

JEL Classification: G11

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1. INTRODUCTION

1.1 Background

With the new supervisory framework coming into effect for Dutch pension funds and insurers, both assets and liabilities will be valued on a marked-to-market basis. The FTK solvency test requires that a 97.5% probability exists that the asset value will exceed the value of the liabilities on a one-year horizon. These fundamental changes, comparing to a previous 4% fixed discount factor, imply that the liability side will be more volatile and controlling down side risk is of great importance for the pension fund to survive. Matching the interest rate risk of assets and liabilities hence becomes more relevant due to the introduction of the marked-to-market valuation and because the solvency test clearly relates the risks of the assets to that of the liabilities.

At the same time, a demand for the limitation of down side risks emerges for passing the solvency test. These risks come from both fixed income assets and equity assets.

Financial institutions can explore solutions to confront this new situation, of which using ultra-long (inflation-linked) bonds; alternatives investment and diversification have been broadly discussed. However, ultra-long bond usually has less liquidity and do not trade frequently. Most alternative investments, such as real estate, do not exhibit a pronounced and stable correlation with pension fund liabilities, which means that the interest rate risk embedded in the long-term liabilities of pension funds is not meaningfully reduced by investing in the asset classes.

Alternatively, increasing contributions and (or) reducing inflation indexations, which have been discussed broadly, are direct ways to maintain pension fund solvency.

This project, other than the methods mentioned above, will concentrate on using derivatives to improve pension fund performance. Derivatives are efficiently suited to minimize risks as they are easily be added into the asset portfolio and having a non-linear payoff. Moreover, they do not dramatically change the existing asset allocations, which is a big advantage since pension funds are willing to retain current asset portfolios that bring risk premiums. Last but not least, down side risk can be efficiently eliminated, which substantially helps pension funds' successfully passing the solvency test. Hence derivatives are ideal components in the portfolio, which can efficiently improve pension fund performances.

Some research has been done about using derivatives in pension fund. In Capelleveen (2004), an investigation about using option in the equity part drawn the conclusion that option can improve the performance and several strategies had been developed and compared in tailor-fashions. Capelleveen (2005) concentrates on nominal interest rate hedging by swaps and swaptions and indicates their usefulness.

1.2 Pension Fund's Assets & Liability profile

First we describe the Assets & Liability Management (ALM) profile. In an ALM balance sheet there are assets portfolio and liability. Liability represents discounted future cash flows of defined benefits, while assets, originally a cumulative amounts of collected contributions, is a portfolio fully invested in fixed income product, equity, derivatives and other financial products. Under FTK, assets and liability are both valued at market prices. Rebalancing takes place after annual contributions are collected and defined benefits are paid, as strategies are determined with disposal of the new collections, for instance, which market and of what amount to invest.

The states and movements of employees, retirees and new comers, form the most essential part of the external environment for the ALM model since the number of individuals in each state and their movements from one state to another substantially influence the cash flows in assets and liability. The Figure 1.1 gives a sight into pension fund:

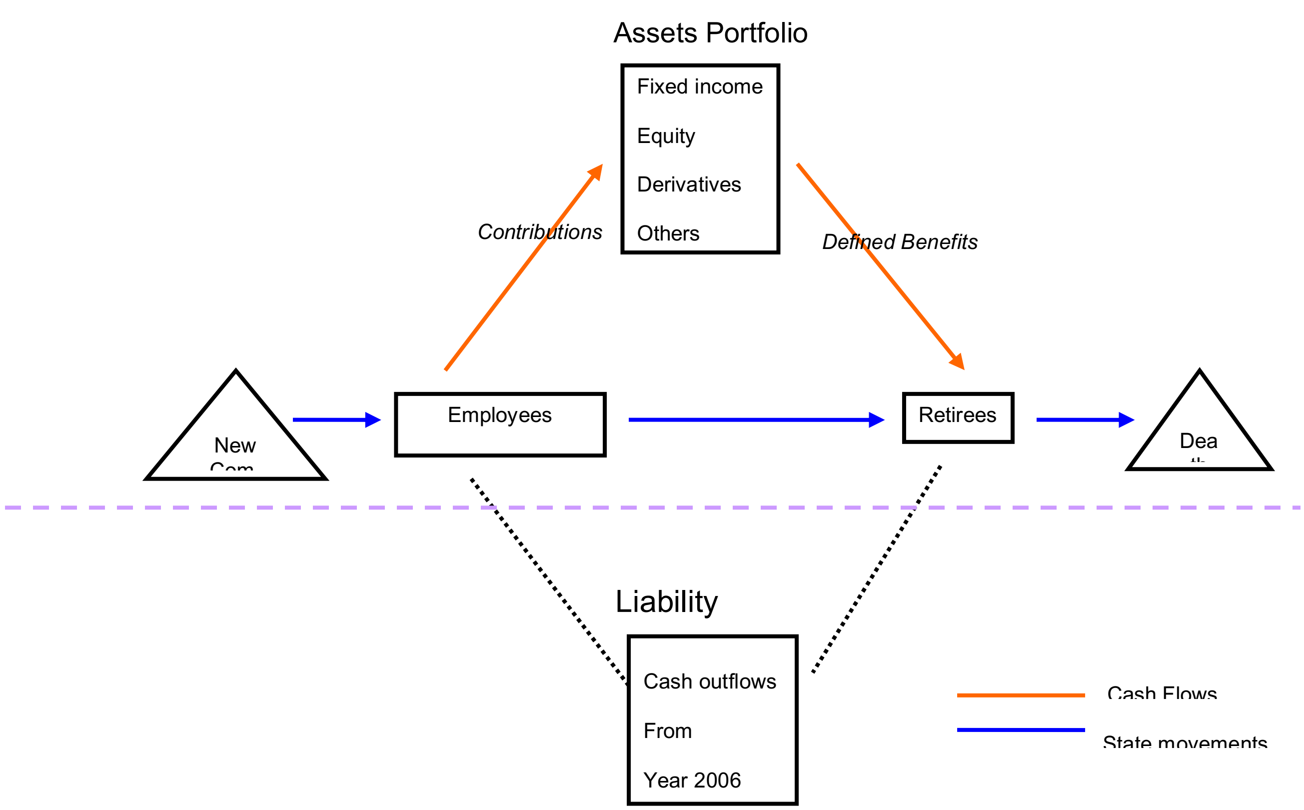


Figure 1.1: the Pension Funds' general profile

1.3 Goal

This project aims at performing in-depth investigation of the risk return profile in pension fund. Relevant derivatives are carefully analyzed and we decide whether and which strategies are the most promising. Then strategies are constructed based on a tailor-fit manner, followed by an evaluation on the expected funding ratio and the probability of being underfunding. Strategies will be modeled and added into the economic scenario generation model for developing a further ALM model specified in chapter 5 and chapter 6, which is a systematic and dynamic asset allocation model.

2 Introductions of Relevant Derivatives

2.1 Risks in Assets and Liabilities

Overwhelming evidences comes for proving the popularity of using derivatives in pension funds for improving risk-return profile for the past years: according to the Bank for International Settlements (IPE.com, 17 November 2006), interest-rate swaps activity surged in the first half of 2006, with analysts putting it down to pension fund interest. The BIS's latest study into over the counter (OTC) derivatives market activity in the first half year also found that "growth in the market for OTC interest rate derivatives accelerated".

Notional amounts of these instruments reached $262trn at the end of June 2006, 24% higher than six months before, there was a particularly high growth in euro-dominated contracts, which were concentrated in interest-rate swaps (28%) and options (29%). There is no doubt about the liquidity of these derivatives. The questions left are: what types of derivatives are available, and in which manners they should be included in the assets portfolio?

In response to these questions, we select two criteria: first, whether adding it can help to improve pension funds' risk and return profile, say, leading to higher funding ratio and lower probability of under-funding; second, whether there is sufficient and liquid market for trading of the selected derivatives instruments. Before deep insights of relevant derivatives, an essential step is to firstly have a big picture of pension fund. Here is a standard balance sheet **table 1.1** for pension fund consisting of both assets and liabilities; it has a current funding ratio 111% (funding ratio= $\frac{Value\,of\ Assets}{Value\,of\ Liabilities}$), which is the typical case in the Netherlands:

Balance Sheet (Starting Point)

Fixed income	900	**Liabilities**	1800
Real Estate	300	**Surplus**	200
Equity	800		
Total	2.000		2000

Funding Ratio = 111%

Table 2.1: A typical Pension Fund Balance Sheet

Assets

To begin with, the assets portfolio includes several popular components: Fixed income securities, Equity and the others. Fixed income securities here refer to any type of investment that yields a regular (fixed) payment. The most common fixed income security that is usually invested by pension funds is bonds. It is simply a promise to pay interest on borrowed money, there is some important terminology used by the fixed income industry:

Principle: Nominal amount being lent.

Coupon: Interest that will be paid in the form of interest rate$\times$Principal.

Maturity: The end of bond, the date that the principal amount should be returned.

Another main component in assets portfolio is Equity. Equity market empirically provides higher return than fixed income security market does.

Fixed income securities and equities, usually compose the most important parts in assets portfolio in pension funds. Risks exist for bond value that yield curve would change with time, and the nominal principle and coupons (if there are any) of bonds would be revalued by spot yield curves at each time node until maturity, which makes the bonds value change with time. As for stocks, stock values are continuously changed with the volatility of equity market. Thus bonds and stocks turn to be the underlying assets needed to be risk-hedged.

Liabilities

The liabilities side of pension fund refers to the current value of future cash outflows to beneficiaries and perspective beneficiaries. A pension fund which collects contributions from active participants (such as employees and employers) would pay distributions to non-active (such as retirees) participants now and in next years as is shown in Figure 1.2 below, this represent the average profile of the pension fund clients of Robeco.

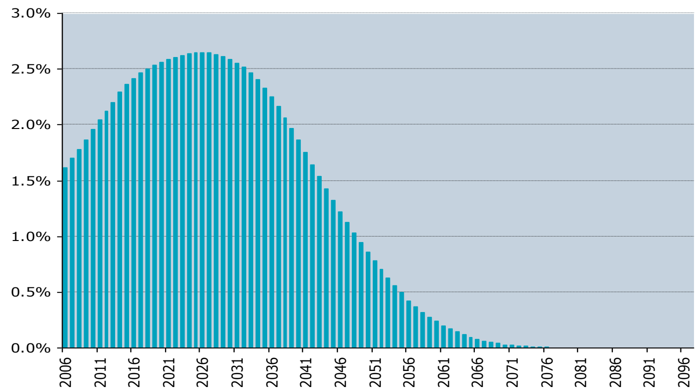


Figure 1.2: Nominal Future cash outflows

For getting the market value of liabilities, all the future cash outflows will be discounted by spot yield curve. For instance, at the start of 2007, all the nominal cash outflows from 2007 on would be discounted by the yield curve at beginning of 2007 to derive the current value of liabilities.

Given nominal cash outflows, an up shifted yield curve leads to a smaller value of liabilities while a down shifted causes an increase in liabilities. Liabilities side faces great interest rate risk which would be hedged by relevant derivatives instruments. In conclusion, there are risks from both assets and liabilities side needed to be hedged. We discuss them right in the following sections.

2.2 Derivatives in Equity

For the asset side, we focus on incorporating financial products into the equity part. Although some pension funds become more and more risk averse, especially confronted with the new coming FTK, and more likely to switch their investment from equity to fixed income products, a fully or largely investing in fixed income products is far from optimal since the equity premiums are missed. The duration of liability in pension fund is usually longer than the average duration of the bonds and other fixed income components in the asset portfolio. The FTK, which requires liability to be valued at a marked-to-market rate, cause substantial decrease in funding ratio if interest rate decreases. Compensation from the risk premium of equity, if it has been added into the asset portfolio with proper tailor-fashion derivatives which efficiently minimize the equity's down side risk, could substantially improve the risk return profile of the assets portfolio, leading to a higher expected funding ratio and a lower probability of under-funding. Meanwhile, investing in equity would bring greater risk to the asset portfolio, we show this in the example below:

Balance Sheet (Equity value -15%)

Bonds	**900**	**Liabilities**	**1800**
Real estate	**300**	**Surplus**	**80**
Equity	**680**		
Total	**1880**	**Total**	**1880**

Funding level = 104%

Balance Sheet (Equity value +15%)

Bonds	**900**	**Liabilities**	**1800**
Real estate	**300**	**Surplus**	**320**
Equity	**920**		
Total	**2120**	**Total**	**2120**

Funding level = 118%

Table 2.2: How equity value changes influence funding ratios

Demonstrated in the tables above, 15% decrease or increase leads to significant changes in funding ratio respectively. With an initial nominal value of equity 800, a 15% value change in one year would cause a nominal change of 120. A decrease of 15% will lead to a equity value of 680 while an increase of 15% will lead to 920.

Underlying assets

At this stage, we simply assume there is only equity and bonds in the asset portfolio. The equity referred here is an ideal combinational product which is supposed to optimally represent the equity part in the asset portfolio, say, an equity index.

Option

Huge volumes of options are traded over the counter by financial institutions, providing a sufficiently liquid option market. The underlying assets are stocks, equity indices, foreign currencies, debt instruments, commodities, and future contracts. Here, the equity included in assets portfolio is equity index. Basically, there are two kinds of options. A call option gives the holder the right to buy the underlying asset by a certain date for a certain price. A put option gives the option holder the right to sell the underlying asset by a certain date for a certain price. For the equity part in the pension fund asset portfolio, a properly constructed option strategy can intuitively help to reduce the risk in equity and decrease the probability of under-funding at the cost of some potential funding ratio growth for the option price. For instance, put options can be added into the asset portfolio to protect the value of equity from going down below a certain level, which can minimize the downside risk. The put option is priced by the Black Scholes model. See Appendix 1

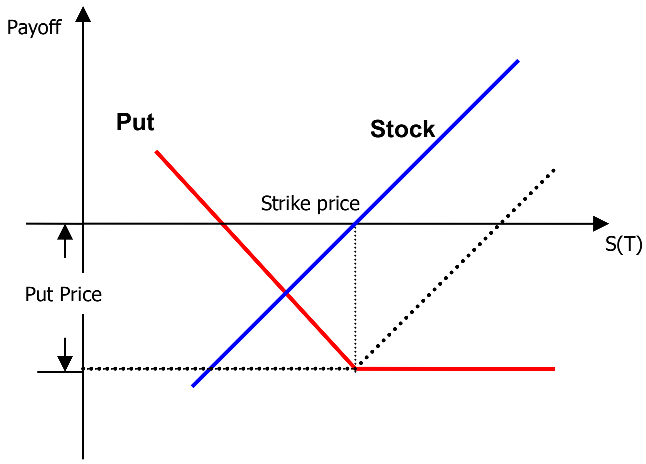


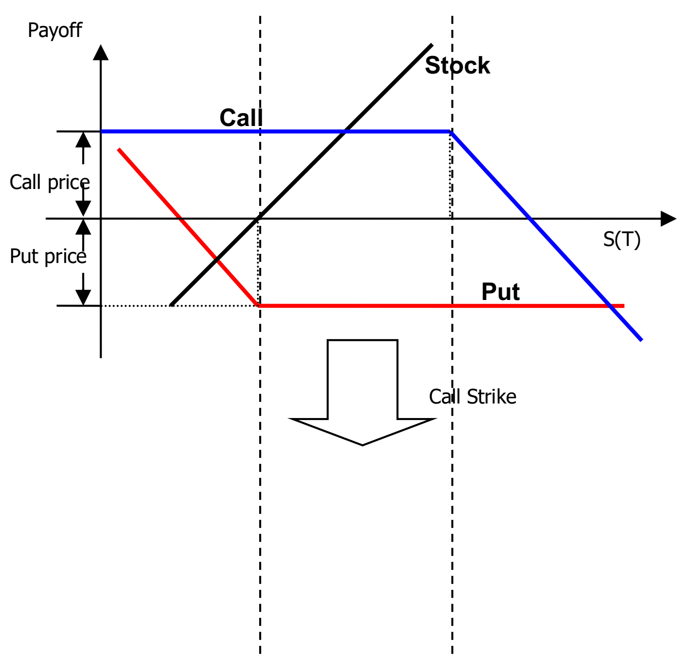
Figure 2.2.1: Put Option's payoff at the time of maturity

As is shown in Figure 2.2, holding stock with 100% put (strike price same as current stock price) protection would be similar with holding a call, which is represented by dashed line in Figure 2.2; In this manner downside risk could be fully controlled, the maximum loss investor would ever experience is the put price.

Collar

Collar is constructed by buying a put option and selling a call option. As is showed above, the collar protects equity value by limiting it in a predetermined range. In our case we consider the case buying put and selling call at the same price to get a self-financing cycle. Theoretically, pension manager constructs a 100% collar with no cost. With collar all the downside risk is eliminated and potential upside profit, however, is sacrificed.

The way collar is constructed is shown below:



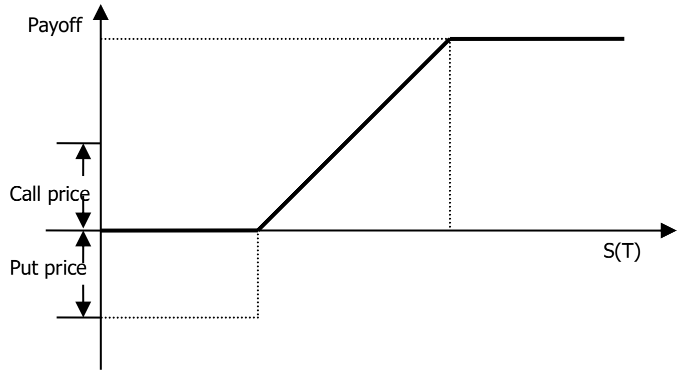


Figure 2.2.2: Collar

2.3 Interest Rate Derivatives

Under the FTK, pension fund liability will be dramatically influenced by interest rate changes since it will be valued in a marked–to-market manner. In this case, an interest rate decrease leads to a lower funding ratio which is not desired. An example:

Balance Sheet (Interest Rate -1%)

Bonds	**949**	**Liabilities**	2080
Real estate	300	**Surplus**	-22
Equity	800		
Total	2049	**Total**	2049

Funding level = 99%

Balance Sheet (Interest Rate +1%)

Bonds	854	**Liabilities**	1558
Real estate	300	**Surplus**	396
Equity	800		
Total	1954	**Total**	1954

Funding level = 125%

Table 2.3: Funding ratios changes with interest rates' influences

A switch from investing in equity to investing in fixed income products is preferred by pension funds under FTK. Holding bonds, especially ultra-long bonds in assets portfolio, can at least partly match the interest rate sensitivity in liability, which can largely investing in bonds supposes to perform as natural risk hedging tool.

However, long term fixed income products could add only limited value to the fund portfolio management. A sufficiently liquid market for fixed income exists usually only for those products with under 10 years' maturities, while the duration of the liability in pension fund is typically longer than 10 years. The mismatch in durations exposes pension fund liability to high interest rate risk.

Meanwhile, certain components in the assets portfolio other than fixed income products, such as stocks, although said to have interest rate sensitivity and valuated by discounting all the future cash inflows, are therefore positively correlated with liability, are far from holding a stable relationship with it in the real world.

Alternatively, swaps and swaptions can more efficiently cut the interest rate risk. Their tremendous popularity supplies a sufficiently liquid market. By including dynamic combination of swaps and swaptions in the assets side, the interest rate risk could be, in theory, perfectly be hedged.

Swaps

Swaps are agreements to exchange in the future cash flows with predetermined conditions. One counter party pays fixed rate on a notional principal, while the other pays floating rate, for instance, Libor, on the same principal for the same period of time.

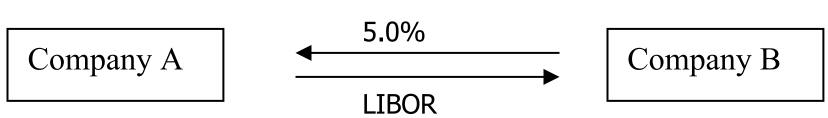


Figure 2.3.1: Swap

The currencies of the two sets of interest cash flows are the same. In the pension fund liability, the fund makes a swap in which it receives fixed payment and paying a floating each year. Here is the cash flows pattern of a swap for pension fund:

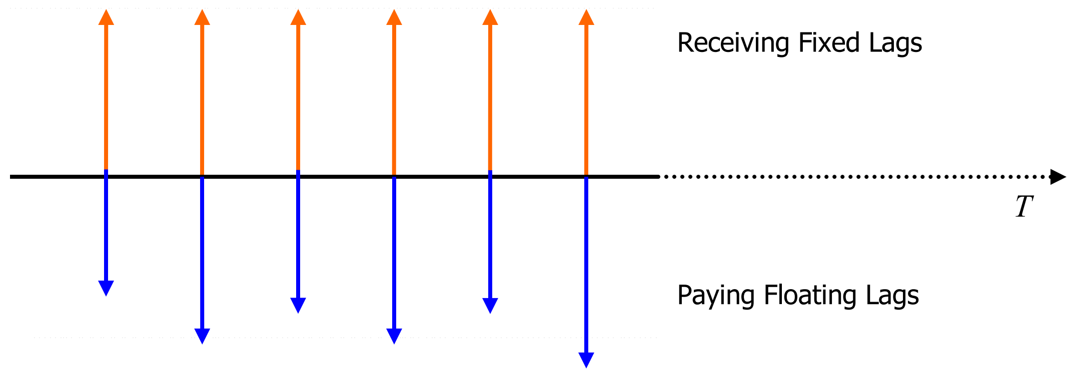
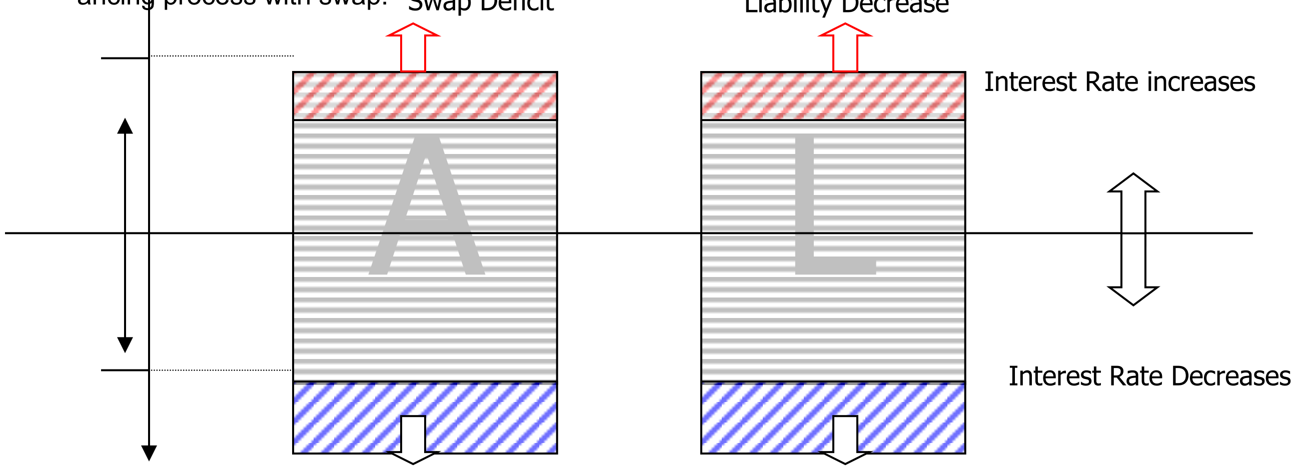


Figure 2.3.2: Cash flows in Swaps

In this manner pension fund receive fixed lags while paying floating. If the interest rate increase then yield curve shifts up, making liabilities value decrease, meanwhile, pension fund pays higher lag which would making assets value decrease; on the other hand, if the interest rate decrease then yield curve shifts down, making liabilities value increase, meanwhile, the pension fund pays lower amount of floating lag than receiving fixed lag, which would making assets value increase. By this way swap functions like a stabilizer to keep the balance of funding ratio. Here is the balancing process with swap:



Refers to the current amount, is the change (negative) after one year under the condition that interest rate increase; refers to the change after one year under the condition that interest rate decrease. From the graph we find that the value Assets (swap involved) and Liability change with the same trends when the interest rate goes up or down, from which we could say, the interest rate risk has been hedged.

Swaptions

Swaptions are options on interest rate swaps. They give the holder the right to enter into a certain interest rate swap at a certain time in the future. In the case that the current interest rates are far below their long term average mean, swaptions are preferred since it can be used to avoid the loss in the swap. Pension fund can easily switch to a swap if interest rate goes up later. The pricing model we used to value swaptions here is the Black Model. (See appendix 2)

3 Economics Scenario Generating
(VAR model)

3.1 VAR model

Vector Auto-regression model is adopted for scenario generating. Value of each economic variable at this stage depends on both weighted value of all the previous variables and the noise.

A VAR model containing selected variables will be built. Including too many variables may harm the model accuracy hence we focus on a few main variables. Selection criteria is set as whether they are main external economic factors which will play main roles in pension fund assets and liability management.

As is mentioned in the last paragraph, including too many variables means there are many parameters to estimate. That is, if we have *n* variables in the model, then *n*n* parameters need to be estimated. A short time horizon starting at some time point from which relevant data is available, could be not sufficient to support a precise estimation. In this case, we only select several main economy factors as variables in VAR model.

3.2 Variables Selection

In this moment six variables are included in the VAR model, which are three interest rate constructors, implied volatility (VIX), inflation and equity premium respectively. These six factors, will build the main external economic environment for further analysis.

Notations in the model vectors are:

$S_{tn} = [R_{tn}, E_{tn}, I_{tn}, V_{tn}]$; t=1,…,T ; n=1,…2000. with

R_{tn} = Yield curve factors in <t,n>
E_{tn} = Returns of asset classes in <t,n>
I_{tn} = Price inflation in <t,n>
V_{tn} = Implied Volatility in <t,n>

In each future time node a number of scenarios are generated by VAR model, consisting of yield curve, implied volatility, inflation and equity return in a future time node t.

Instead use real market rate, we use the method described by Nelson Siegel to get the entire curve (See Appendix 3).

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda_t \tau}}{\lambda_t \tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda_t \tau}}{\lambda_t \tau} - e^{-\lambda_t \tau}\right) + \varepsilon\ ,$$

Data sets are yields with maturity of 3, 6, 12, 24, 36, 60, 84, 120, 240, 360 months. At each time point a series of yields (of 3, 6, 12, 24, 36, 60, 84, 120, 240, 360 months) are collected and set as dependent variable Y, the corresponding series of independent variables X are listed below:

	3M	6M	12M	24M	36M	60M	84M	120M	240M	360M

β1's Loading	1	1	1	1	1	1	1	1	1	1
β2's Loading	0.915	0.840	0.714	0.531	0.411	0.271	0.198	0.139	0.070	0.047
β3's Loading	0.080	0.142	0.226	0.293	0.294	0.243	0.191	0.139	0.070	0.047

Table 3.2: Loadings

According to $Y = \beta' X + \varepsilon$, β_{1t}, β_{2t} and β_{3t} are OLS regressed for that specific time point. With the same process β_{1t}, β_{2t} and β_{3t} could be derived for each time point between 1983 Q1 to 2006 Q3.

3.3 Data collection

Historical data are required to estimate the VAR model. Times series of data are selected and adjusted in a quarterly manner during 1983 quarter one to 2006 quarter three. Equity return is quarterly return while Inflation is annualized. Quarterly data seems to be good compromise between annual and monthly data. We have 95 quarterly observations available while only 24 yearly observations, which could hardly support the coefficient estimation. Monthly data, according to Hoevenaars, Molenaar and Steenkamp (2003), are comparatively noisy and consequently seem less appropriate for capturing long-term dynamics.

The data are collected from 1983 mainly because only since then the interest rates have been controlled reasonably by the appointment of the 12th president of the US Federal Reserve, Paul Volcker. When making forecasting about future it is believed that the current policy will keep working in long term thus we collect data from 1983.

Yield curve

Firstly, the yield curve data are collected. Quarterly US data have been collected from *Board of Governors of the Federal Reserve System*. Given the availabilities, we select annualized yield data with maturities 3, 6, 12, 24, 36, 60, 84, 120, 240, and 360 months during the time period 1983-2006. According to Nelson Siegel model, the β_{1t}, β_{2t} and β_{3t} will be derived by regressing these yield data to the given loadings (details are describe in above ***Nelson Siegel Model***). In this case, three series of β_{1t}, β_{2t} and β_{3t} with a quarterly time horizon 1983-2006 are available.

We get various yield data for different time periods in 1983 Q1 – 2006. The availabilities of data for each period are given below:

1983 Q1 – 2001 Q4 3, 6, 12, 24, 36, 60, 84, 120, 240, 360 (months)
2002 Q1 – 2005 Q4 3, 6, 12, 24, 36, 60, 84, 120, 240 (months)
2006 Q1 – 2006 Q3 3, 6, 12, 24, 36, 60, 84, 120, 240, 360 (months)

With the yield data given from these periods, we derive β_{1t} , β_{2t} and β_{3t} for each time point. As is shown above, the data set between 2002 Q1 and 2005 Q4 do not contain 360 month yield data. In this case, we do regression for these time points only with the available data and relevant loadings in the table 3.2.

Implied Volatility

Implied volatility shows the market's expectation of 30-day volatility. It is constructed using the implied volatilities of a wide range of S&P 500 index options. This volatility is meant to be forward looking and is calculated from both calls and puts. The VIX is a widely used measure of market risk. The data for VIX are original from Chicago Board Option Exchange (CBOE) and we got them from DataStream. Take into account that only data from 1986 are available, we estimated implied volatility by calculating the historical volatility for 1983-1986. The method for deriving historical volatility is first calculating monthly standard deviation for S&P 500 daily returns. Then annualized these monthly standard deviations through multiply it by $\sqrt{12}$, in which manner annual standard deviation (volatility) could be derived.

$$\sigma_t = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(r_{t-1} - \mu_t)^2} \times \sqrt{12}$$

With $\mu_t = \frac{1}{N}\sum_{i=1}^{N} r_{t-i}$

Here N refers to the average number of days in a month.

To check if it is proper to represent VIX by historical volatility, we test the correlation between VIX and Historical Volatility (0.766) and plotting graph below:

(Red line represents historical volatility and blue represents VIX. From the correlation and graph we found that the historical volatility could be roughly used as indicator of VIX).

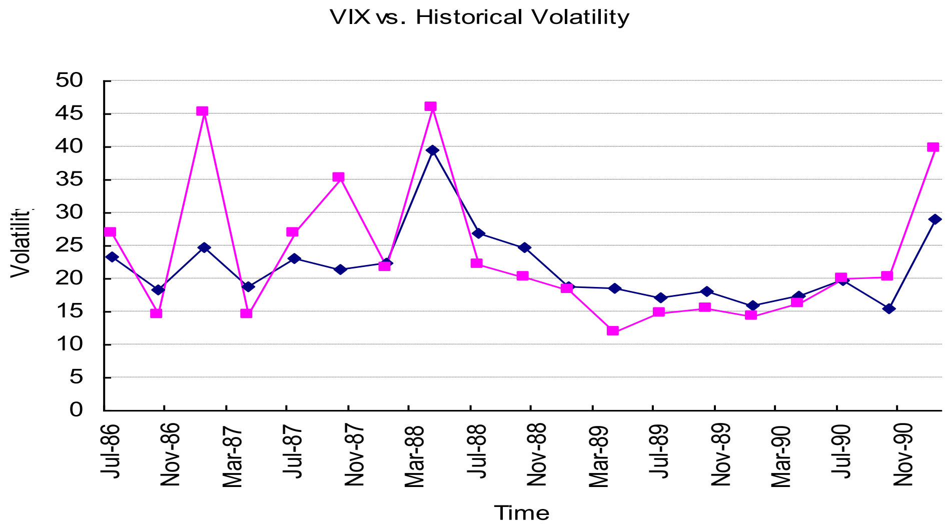


Figure 3.3: VIX vs Historical Volatility

Equity Premium

The US equity premium data are from *Ibbotson Associates*. We quarterized the original monthly data. For instance, the monthly equity returns for 1983 quarter one are i_1, i_2 and i_3. Then the quarterly equity return is $i= (1+i_1) *(1+i_2)*(1+ i_3) – 1$. By this way a series of quarterly data are derived for the period from 1983 to 2006. Then for getting premium, we subtract $\frac{1}{4} y_t(0) = \frac{1}{4}(\beta_{1t} + \beta_{2t})$ from equity return. β_{1t} and β_{2t} are the instantaneous regressed results.

Inflation

The original monthly Inflation data are from *Ibbotson Associate.* We annualized these monthly data use the same method for equity premium but on a yearly base.

3.4 Parameter Estimation

We use the VAR model to construct scenarios of the future environment. In a VAR model, the value of each year's object in a vector depends on linear combination of object values from previous time point in a multidimensional manner:

$$(Y_t - \mu) \sim N(\sum_{p=1}^{P} \Phi_p * (Y_{t-p} - \mu), C).$$

Here μ refers to vector of average value for the six factors during the given time horizon; Φ_p refers to a matrix of coefficients for independent variable $(Y_{t-p} - \mu)$; *p* is the order of lags; *C* is a matrix of noise.

When building the VAR model, Returns of asset classes, Price inflation are observed variables. Yield Curve, however, will be characterized by three variables β_{1t}, β_{2t} and β_{3t}, which capture most of information of the yield curves.

There are several popular methods applied for parameter estimation, for instance, Ordinary Least Square (OLS) and Yule Walker.

We first take insight to OLS. Based on the data given (details in chapter 3.3), the estimated coefficient matrix is given below:

The red numbers are the t-statistics for these coefficients. The adjusted R square and F-statistics are included in the table.

	Mean	β2-μ	β3-μ	β1-μ	VIX-μ	Inflatio-μ	Equity premium -μ	Adjusted R^2	F-statistics
β2-μ	-0.75	**0.72**	0.12	0.15	0.01	-0.01	0.02	0.67	33.14
		7.62	1.05	1.45	0.46	-0.06	1.34		
β3-μ	-2.36	**0.19**	**0.78**	**-0.17**	-0.01	0.05	0.00	0.86	99.52
		4.21	13.84	-3.43	-0.89	0.76	-0.48		
β1-μ	7.44	-0.03	0.05	**0.99**	-0.01	-0.05	0.00	0.94	256.6
		-0.65	1.09	24.06	-0.64	-0.80	-0.38		
VIX - μ	18.4	0.02	0.38	-0.14	**0.65**	-0.34	0.08	0.42	12.44
		0.04	0.72	-0.31	7.14	-0.53	1.02		
Inflation-μ	3.12	0.04	0.01	0.03	0.00	**0.76**	0.00	0.65	29.82
		0.72	0.16	0.50	-0.30	9.73	-0.39		
Equity premium - μ	1.71	-0.98	1.14	1.04	0.18	-0.79	-0.02	0.01	1.09
		-1.60	1.49	1.59	1.37	-0.85	-0.16		

Table 3.4.1: OLS estimated results

As we can see in above table, the bold faced numbers are significant estimated coefficients at 90% level. Here are the explanations: for (β2- μ), the only significant coefficient comes from previous (β2- μ), which means that it is not significantly influenced by other factors. The coefficient 0.715 indicates that there is significant autocorrelation for the short-term factor β2.

For (β3- μ), significant coefficients come from previous (β2- μ), (β3- μ) and (β1- μ), all the three previous constructive factors have significant influence for current (β3- μ). The coefficients 0.190, 0.779 and -0.166 indicate that the strongest decisive power comes from autocorrelation, with a positive influence from short-term factor and a negative one from long-term factor.

For (β1- μ), the only significant coefficient comes from previous (β1- μ), which also means that it is not significantly influenced by other factors. The coefficient 0.994 indicates that there is strong autocorrelation for long-term factor β1.

Also for (VIX- μ) and (Inflation- μ), the only significant decisive influence comes from strong autocorrelations. The coefficients 0.645 and 0.759 indicate that they have comparatively strong link with their previous values, respectively.

There are no significant coefficients for (Equity premium- μ), however, according to the table of regression results. Common sense tells that the equity market is one of the most volatile markets thus it is of great probability that there is weak autocorrelation between equity premiums.

Adjusted R square and F-statistics

We notice that the adjusted R square for (Equity premium- μ) are especially low, which indicates that there is only very small proportion of the variation in the dependent variable accounted for by the explanatory variables.

F-statstic tests whether all the coefficients in the model are equals to zero. A large F-statistics value provides that not all the coefficients in the model are zero. From table 1.4.1 we could draw the conclusion that only the F-statistic for (Equity premium- μ) is low enough to accept the hypothesis that all coefficients equal to zero, equity premium is independent of all the possible explanatory factors.

	β2	β3	β1	VIX	Inflation	Equity Premium
β2	**1.30**	0.00	0.26	-1.78	0.07	1.62
β3	0.00	**0.30**	-0.14	-0.40	-0.01	1.01
β1	0.26	-0.14	**0.22**	-0.22	0.10	-0.45
VIX	-1.78	-0.40	-0.22	**26.41**	-0.05	-22.26
Inflation	0.07	-0.01	0.10	-0.05	**0.39**	-0.44
Equity Premium	1.62	1.01	-0.45	-22.26	-0.44	**55.62**

Table 3.4.2: Noise matrix for Table 3.4.1

The noise matrix is covariance matrix of the six factors involved in the model. The bold faced numbers in table 3.4.2 refer to the variances of each factor. For instance, the variance for equity premium is 55.6234 leads to a standard deviation as squared root of 55.6, which are 7.5. Since it is the quarterly volatility, annual volatility could be derived as $\sigma_t = 7.5 \times \sqrt{4}$ =15.0.

Debates existed that whether we should drop these insignificant coefficients. Here we test if excluding these coefficients will significantly influence the estimated results: at a 90% level, we drop these independent variables which are not significant enough. Then the new coefficient matrix becomes:

	Mean	β2-μ	β3-μ	β1-μ	VIX-μ	Inflation -μ	Equity premium -μ	Adjusted R^2	F-statistics
β2-μ	-0.75	0.823	-	-	-	-	-	0.675	196
		14	-	-	-	-	-		
β3-μ	-2.36	0.192	0.789	-0.14	-	-	-	0.865	201.4
		4.347	15.412	-3.332	-	-	-		
β1-μ	7.44	-	-	0.967	-	-	-	0.944	1597
		-	-	39.97	-	-	-		
VIX – μ	18.4	-	-	-	0.652	-	-	0.431	72.14
		-	-	-	8.493	-	-		
Inflation-μ	3.12	-	-	-	-	0.822	-	0.656	180.2
		-	-	-	-	13.43	-		
Equity premi-um – μ	1.71	-	-	-	-	-	-		
		-	-	-	-	-	-		

Table 3.4.3: OLS estimations without Equity Premium

With new noise matrix:

	Beta 2	Beta 3	Beta 1	VIX	Inflation	Equity Premium
Beta 2	1.37	-0.01	0.26	-1.70	0.07	1.83
Beta 3	-0.01	0.31	-0.14	-0.43	-0.01	0.92
Beta 1	0.26	-0.14	0.22	-0.20	0.10	-0.43
VIX	-1.70	-0.43	-0.20	27.48	-0.06	-22.14
Inflation	0.07	-0.01	0.10	-0.06	0.40	-0.50
Equity Premium	1.83	0.92	-0.43	-22.14	-0.50	59.76

Table 3.4.4: Noise matrix for 3.4.3

From the first sight, we found that from the F-statistics that excluding insignificant coefficients giving a more accurate model, since larger F-statistic value implies more strength of explaining power of the model. No big difference between each pair of adjusted R squares.

When deciding which model, including or excluding insignificant coefficients, to be adopted as our VAR model, we also check the fitting graphs of these 6 factors:
Blue lines: Observed data; red lines: Fitting line based on the model with all the coefficients including the insignificant ones; Yellow line: Fitting lines without insignificant coefficients.

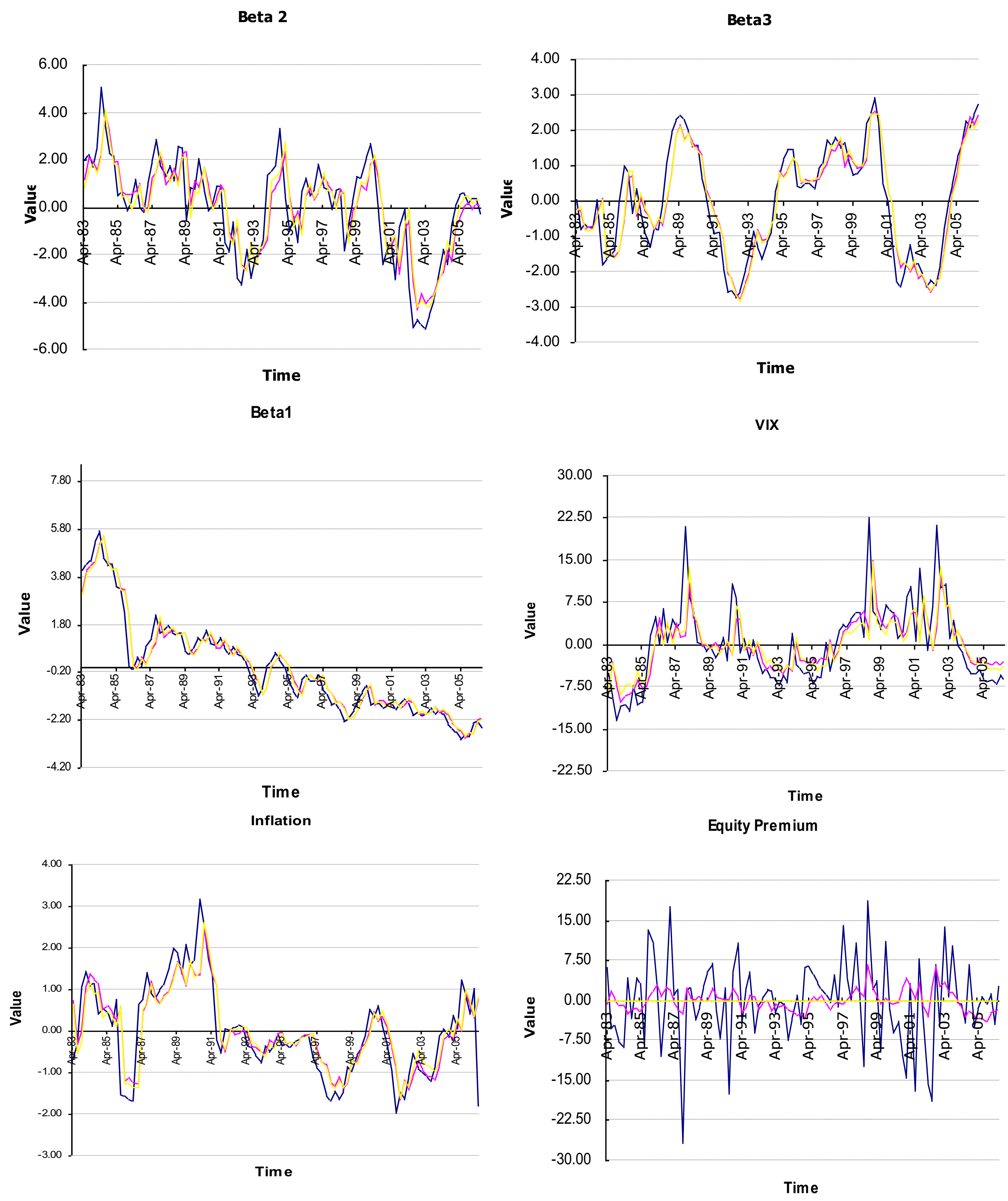


Figure 3.4.1: Fitting lines with or without insignificant coefficients

As we can see from the graphs, for the first five plots, with or without insignificant coefficients in the estimated model would not cause obvious differences in the fitting lines. That is, including or excluding insignificant coefficients would cause slight difference when creating scenarios. However, in last plot we could find an obvious difference between these two fitting lines. As we can see in the plot, the maximum difference was up to roughly 7 percent in some quarter 1998, concerning that it is quarterly equity premium, the difference between annualized equity premiums is roughly 28 percent. When making forecasting of equity premium for the next years, we expected a comparatively stable mean value around historical mean 1.71 percent per quarter, since it is weak-linked with previous performance. Adopting model with all the coefficients would harm this stability, as it is shown in the last plot; it would add positive or negative drifts to cause the expected equity premium to be too volatile. Given the results we got above, we decide to continue with the estimated model with all the coefficients for Yield curve, Inflation, VIX; while for equity premium, each coefficient equals to zero.

After picking the estimated model, we need to decide which method to use for coefficient estimation. Boender, Dert, Heemskerk and Hoek (2003) take the Yule Walker method to estimate coefficients. In Hoevenaars, Molenaar and Steenkamp (2003), OLS is adopted. While according to Neumaier Schneider [2001], as a computationally efficient method of estimating the parameters of AR models from high-dimensional data, a stepwise least squares algorithm is proposed. This algorithm computes model coefficients and evaluates criteria for the selection of the model order stepwise for AR models of successively decreasing order. Neumaier and Schneider (2001) discuss properties of the stepwise least squares algorithm and compare this algorithm with other methods for the estimation of AR parameters.

In the stepwise method, the optimum order p of this VAR model is generally chosen as the optimizer of an order selection criterion (Lütkepohl 1993, Chapter 4). The order selection criteria implemented here are Akaike's (1971) Final Prediction Error (FPE) criterion and Schwarz's (1978) Bayesian Criterion (SBC). Lütkepohl (1985) compared these and other order selection criteria in a simulation study and found that Schwarz's Bayesian Criterion chose the correct model order most often which leads, on the average, to the smallest mean-squared prediction error of the fitted VAR models. Schwarz's Bayesian Criterion is therefore the default order selection criterion.

A comparison among OLS, Yule-Walker and Stepwise are in Appendix 4.

3.5 Evaluation

We check the estimated autocorrelation function of the corresponding residuals based on step least square method, Figures 3.5.1 display the correlogram of the corresponding residual sequence.

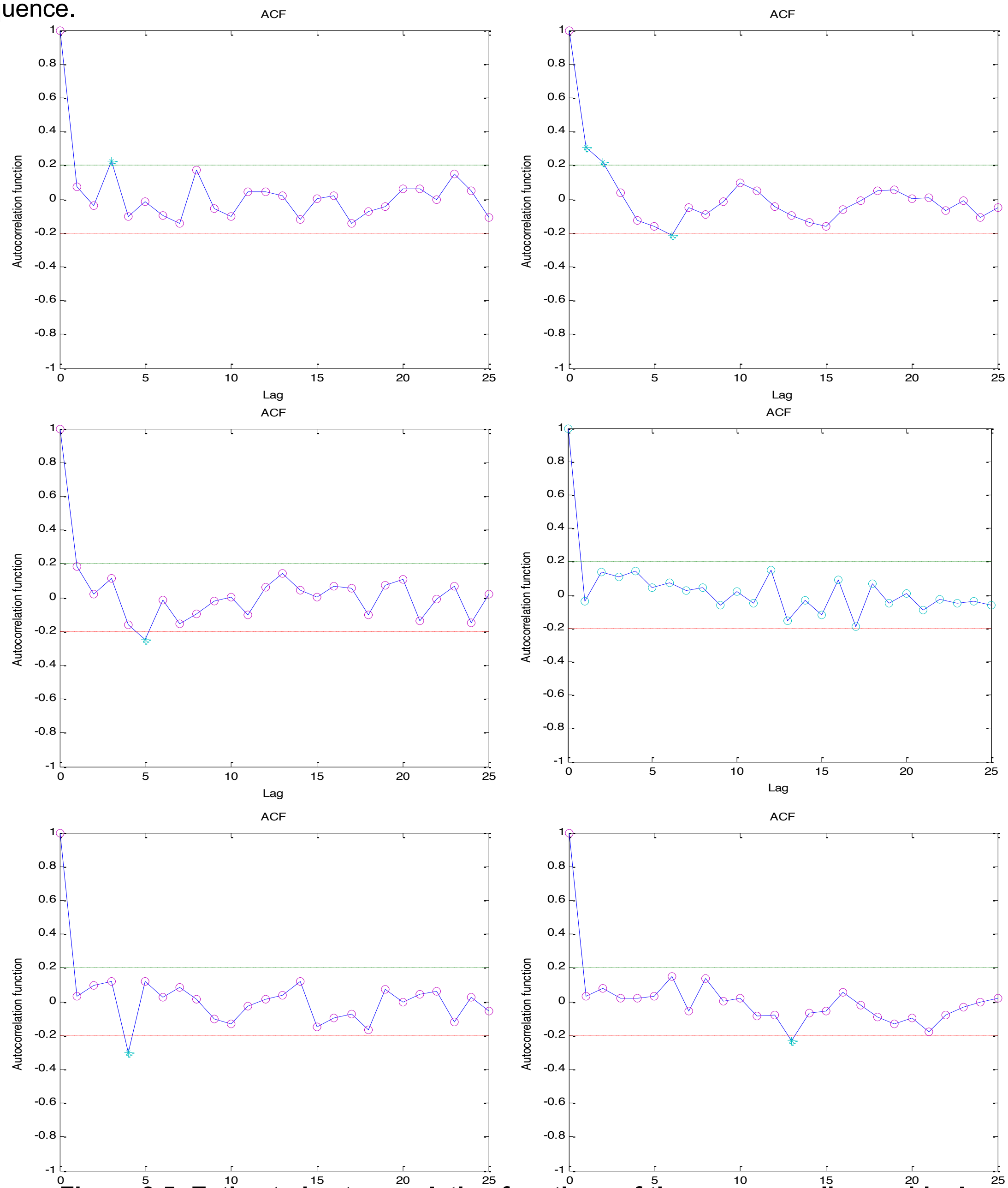


Figure 3.5: Estimated autocorrelation functions of the corresponding residuals

From this we find few reasons to doubt that the residuals resemble a white noise process, as they should.

3.6 Simulation

The simulation contains 2000 scenarios for the next 25 years. It is based on the estimated coefficient matrix and noise matrix, start point is set as [0.0715 2.7910 -0.7179 -6.4027 -1.1226 3.2939] as result of excess of current β_{2t} , β_{3t} , β_{1t} , VIX, Inflation and Equity premiums value on historical mean (details given in the next paragraph).

Modification of the model

As mentioned before, the basic model for estimation is showed as following:

$$(Y_t - \mu) \sim N(\sum_{Y_{p=0}}^{P} \Phi_p * (Y_{t-p} - \mu), C).$$

Where μ refers to a vector which contains historical mean of β_{2t} , β_{3t} , β_{1t} , VIX, Inflation and equity premium. Given data from 1983 Q1 to 2006 Q3 we derived [-0.75 -2.36 7.44 18.40 3.12 1.71] as historical mean, among which β_{2t} and represent influences from short term and medium term while β_{1t} is simply the historical long term yield level according to Nelson Siegel Model. From the vector we get the mean for long term factor β_{1t} 7.44%, which, based on current external economics situation, seems to be too high to be accepted as estimation for future long term yield. Alternatively, according to popular forecasting from different institutes or economists, we set a mean for β_{3t} as 5.5% which is a more approximate and reasonable estimation for future long term yield. And as required by Dutch regulator, the expected equity premium should not be larger than 0.75% per quarter. That is, we replace [-0.75 -2.36 7.44 18.40 3.12 1.71] with [-0.75 -2.36 5.50 18.40 3.12 0.75] as the mean vector μ in our model. The last three components in the vector, 18.40% and 3.12% refer to mean values for quarterly annualized VIX and inflation, respectively.

We set the starting value as current [Beta2, Beta3, Beta1, VIX, Inflation, EquityP] – Forecasting Mean, where current situation is set to be [-0.68 0.43 4.78 12.00 2.00 5.00] as derived from spot yield curve and economics environment.

	Beta 2	Beta 3	Beta 1	VIX	Inflation	EquityP
Historical Mean	-0.75	-2.36	7.44	18.40	3.12	1.71
Forecasting Mean	-0.75	-2.36	5.50	18.40	3.12	0.75
Start Value	0.07	2.79	-0.72	-6.40	-1.12	4.25

Table 3.6.1: Starting Values

Simulation: Yield Curve

Yield curves in any future time point are got based on the simulation for β_{2t} , β_{3t} , β_{1t} . Since the yield couldn't be below zero, we put a lower constraint for yield during the process of scenarios generating. Based on the model described above 2000 scenarios of yield curve in 2032 (25 years from now) are created. See the graph below:

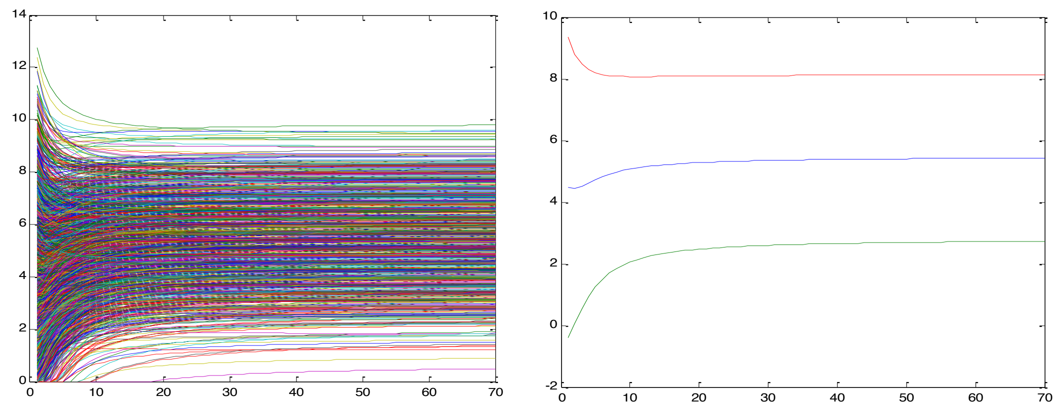


Figure 3.6.1: Simulated yield curves after 25 years, right with 95% confidential intervals

The left graph describes the 2000 scenarios and the right represents mean yield curve of these 2000 curves. From them we find that after 25 years, in coincidence with the forecasting now, the long term yield will be between 5 percent and 5.5 percent, while the short term will be around 4.5 percent, which is less than the long term yield. The upper and lower lines give a 95% confidence interval for the yield curve.

Simulation: Bond Return

Based on the yield curves we could get bond return. We assume a 10 years 5 percent coupon bond with nominal value of $1000, the bond is issued at the start of 2031. Then bond return is calculated by comparing the real price of bond at start of the year 2032 to the real price at the start of the year 2031.

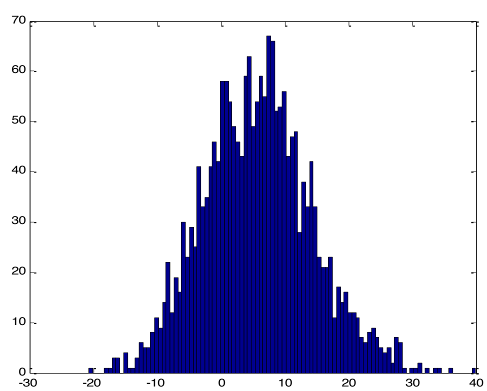


Figure 3.6.2: Histogram of bond returns after 25 years

Based on the calculated bond return we get above graph, which tells the distribution of the 2000 estimated bond return. They have a mean of 5.664 percent and a standard deviation of 8.6638 percent. The 7-10 years duration US treasury has a standard deviation of 7.85 percent for roughly the same time period.

Simulation: VIX

The histogram for 2000 simulated annual inflation in 2032 Q4 is showed here:

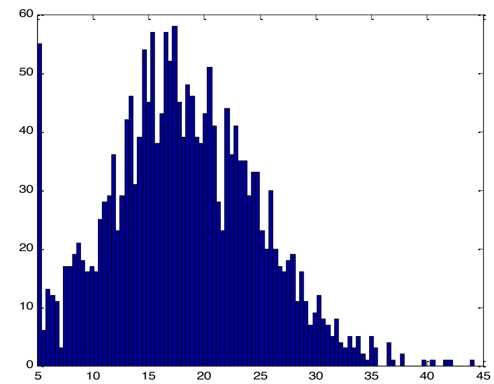


Figure 3.6.3: Histogram of Implied Volatility after 25 years

We put a constraint on as lower boundary of VIX: 5%, which is roughly the historical lowest VIX. The simulated VIX in 2032 has a mean of 18.178 percent and with a standard deviation of 6.61 percent. Further comparison with historical data results will be given later.

Simulation: Inflation

The histogram for 2000 simulated annual inflation in 2032 Q4 is showed here:

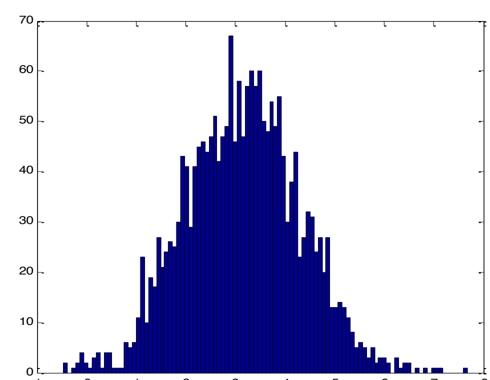


Figure 3.6.4: Histogram of Inflations after 25 years

With a mean of 3.151 percent, roughly in coincidence with the historical mean for inflation: 3.123, and with a standard deviation of 1.17 percent.

Simulation: Equity Premium

The histogram for 2000 simulated annualized equity premiums in 2032 Q4 is showed here:

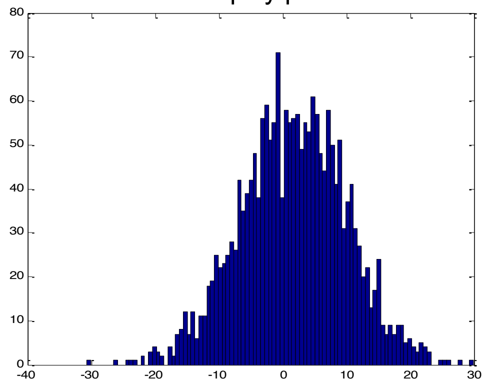


Figure 3.6.5: Histogram of Equity premiums after 25 years

With a mean of 1.884 percent, and a standard deviation 8.205 percent; they are roughly in coincidence with the historical mean for equity return: 1.710, and with the prediction variance 61.8.

Simulation: Correlations

The correlation between bond return and equity premium from simulation of year 2032 is 0.0857, between bond return and inflation is -0.1068, between VIX and equity premium is -0.356, inflation and equity premium is -0.087.

Compared to historical correlations of corresponding pairs: Bond return and equity premium 0.05, Bond return and Inflation -0.078, VIX and EP: -0.30, Inflation and EP -0.11.
Here is the table for comparison:

	Historical Correlation	Simulated Correlation
Bond return and Equity Premium	0.05	0.09
Bond return and Inflation	-0.08	-0.11
VIX and Equity Premium	-0.3	-0.36
VIX and Inflation	-0.11	-0.09

Table 3.6.2: Historical correlations vs Simulated Correlations

4 Short term analysis

4.1 Profile

Short term analysis for pension fund is conducted based on the scenarios we simulated. Reason for doing short term analyses before in-depth long term investigation are two basis: firstly, for simplicity, derivatives strategies will be constructed and evaluated on a one-year base; then suitable strategies are selected for further long term analysis. Second, the coming FTK asks for solvency test which requires with 97.5% probability the funding ratio will be above certain level after one year.

Based on a one year horizon, the asset and liability of pension fund will be modeled separately and combined later to test the funding ratio and probability of under-funding. This short term analysis will be base on a closed pension fund, where no contributions are collected any more and with known future nominal cash outflow at the end of the year. That is, after one year, there will be only a predetermined cash outflow from the asset portfolio, and the liability will decrease by the corresponding nominal amount. For the following analysis, we concentrate on the derivatives strategies in both asset and liability sides.

State movements are not included in this short term analysis. The ongoing research based on three closed pension funds, with starting funding levels 120%, 110% and 100% representing optimistic, neutral and pessimistic pension fund situations.

For long term analysis, ongoing pension funds, instead of closed pension funds, will be investigated. State movements, together with changing distribution and collection levels, which cause various cash inflows and cash outflows, will be modeled.

All the three pension funds are assumed to have same assets portfolio construction (with same components and same percentage of each asset), same liability (with the same patterns and amounts of cash outflows). As in the closed pension fund, there is no new collections, which means no new cash inflows in the future. Each assets portfolio simply contains two parts, fixed income products (100% 7 years zero-coupon bond, as usually the duration of fixed income in Dutch pension fund is roughly 6-7 years) and equity. The asset portfolio will be re-evaluated experience value change in one year according to the external economic environment, and there will be a cash outflow at the end of one year as distributions. There will be a corresponding reduction in the liability side, and the rest of future cash outflows will be rediscounted by the spot rates at the end of one year. Below is the graph of expected situations on a one year horizon. Blue pots represent the risk-return profile with initial funding ratio 100% while yellows and reds represent those of initial funding ratios of 110% and 120%, respectively.

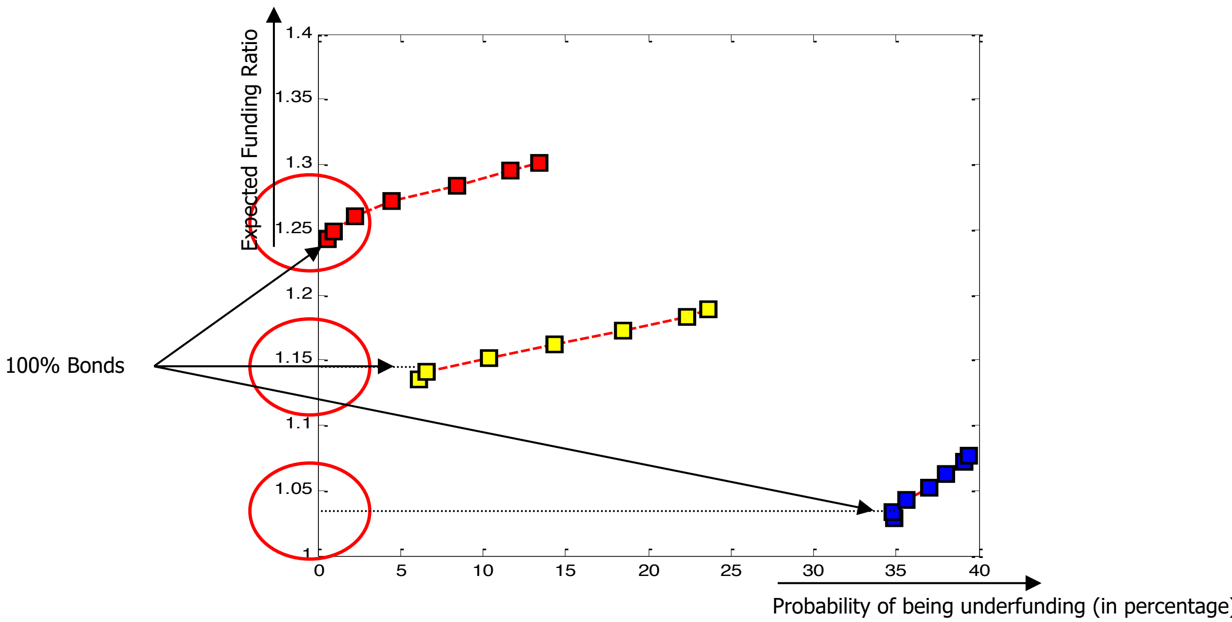


Figure 4.1.1: Funding ratios and Probabilities of underfunding

It could be showed from Figure 4.1.1 that with the increased percentage of equity, at all the three levels, the expected funding ratio increases with probability of under-funding. It could be explained by that equity assets have higher return and also higher risk comparing to fixed income assets. Since the current yield curve is lower than historical average, while simulated yield curve for next year will be closer to historical average and cause a shifting up of funding ratio (with 100% bonds), which we could see above.

To investigate how much is the influence to risk-return profile from yield curve, we do the following test: keep next year's mean yield curve unchanged as this year's yield curve, while adding volatility as it should be. In this case, next year's risk-return performance will theoretically mainly be influenced by equity value changes. We get the following graph:

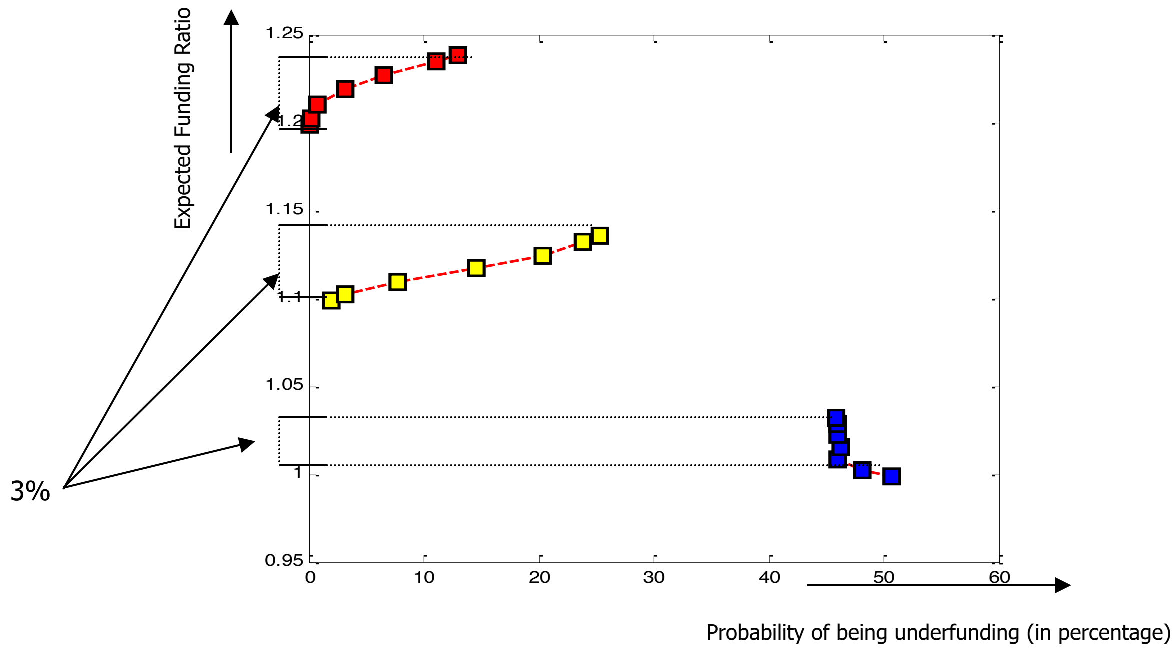


Figure 4.1.2: Funding ratios and Probabilities of underfunding in smooth interest rate environments

From the Figure 4.1.2 we find that since yield would keep unchanged, the funding ratios with 100% bonds in assets portfolio will remain what they are at all the three initial funding ratios: 100%, 110% and 120%. As is showed, starting from funding ratio 100% with full bonds roughly leads to a 50% probability of under-funding because of the added volatility.

With the increase of percentage of equity, for funding ratio 110% and 120%, the probability of under-funding and the funding ratio will increase. While for funding ratio 100%, funding ratio will increase and probability of under-funding will decrease. For every initial level, the difference of funding ratio between 100% bonds and 100% equity is always about 3%, which is the predetermined equity premium as demonstrated in Chapter 3, VAR modeling part.

In the following graph 4.1.4, the comparison is conducted between two situations (natural case or no-yield curve change case) of pension risk-return profiles with initial funding ratio 110%. As is showed in the graph, with 100% bonds in assets portfolio, a simulated yield curve change would cause 3.72% difference in expected funding ratio, implies that 3.72% funding ratio increase comes from an upward shifting yield curve. For 100% equity case, we gain 5.5% funding ratio increase by upward shifting yield curve. In both cases liability becomes less since future cash outflows are discounted by higher yields, while duration gap between assets and liabilities for the 100% bonds case is larger than that of the latter case because bonds asset partly offset the duration of liability. Hence when yield curve shifts up, expected funding ratio of 100% equity pension will be higher than 100% bonds pension.

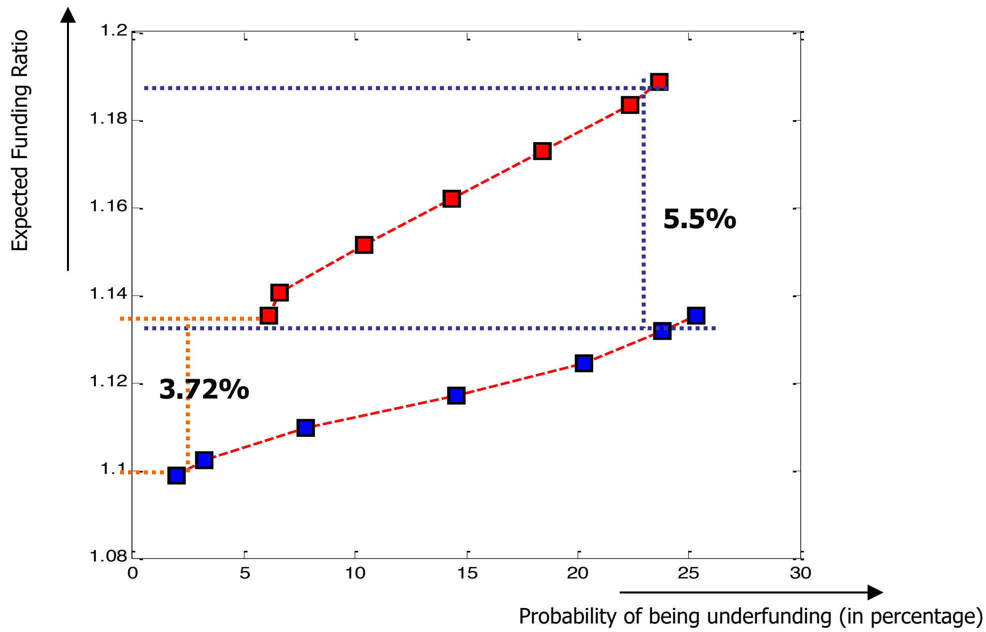


Figure 4.1.3: Comparisons of Funding ratios and Probabilities of underfunding: Increasing vs Smooth interest rate environments

4.2 Derivatives for Equity Implementation and Valuation

Put and call option is being investigated to be incorporated into asset portfolio. These options provide protection on the portfolio from big loss caused by equity prices dropping. As given a strike price, the value of the portfolio can be hold above certain level hence the down side risk can be eliminated.

We use Black Scholes to price the put and call option.

4.2.1 European Option:

a. Put (100% Protection at the money)

First a 100% protection for equity in the asset portfolio is being considered and implemented. Based on the research, whether including 100% put would improve risk-return profile depends on the cost of option; currently at the money put is priced at 4.45, with VIX 17%, spot and strike price both at 100, time 1 year and annualized risk free rate 5%. Within these puts, we got Figure 4.2.1 shown below:

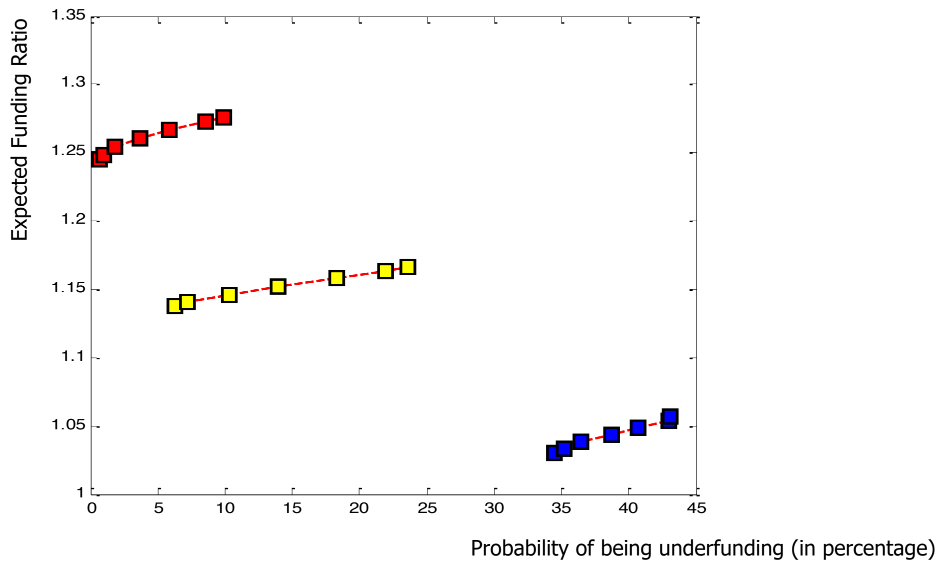


Figure 4.2.1: Expected funding profiles for funds with three initial funding levels (Put)

We compare each pair of risk-premium profiles of the same initial funding ratio while with different derivatives strategies, say, 110% put protection or no derivatives involved. For initial funding ratio 110%, red pots demonstrate the no-derivative case while blue pots refer to the case with 110% at the money put.

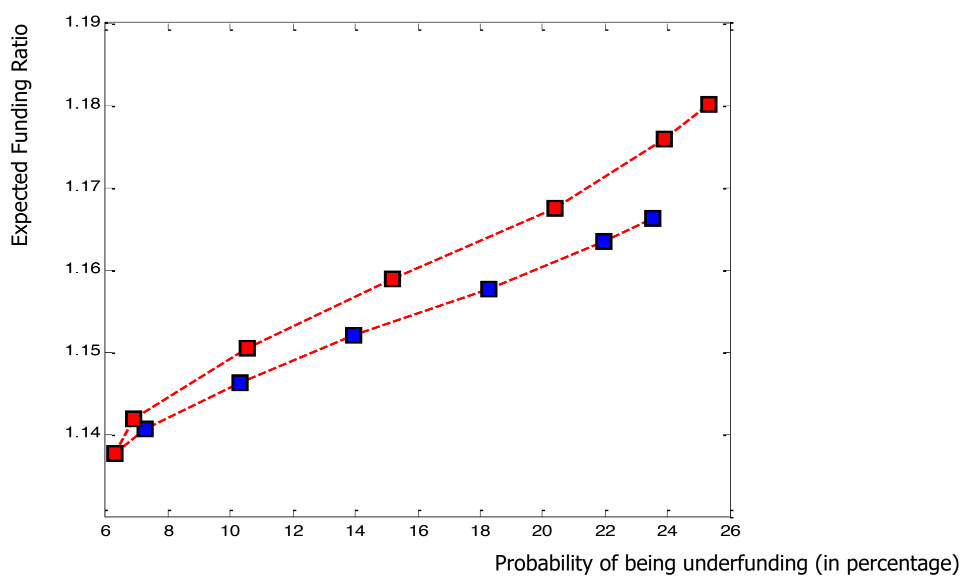


Figure 4.2.2: Comparison between non-derivatives and put-involved funds (initial ratio 110%)

It could obviously be found from the graph the put strategy could reduce risk of being underfunding, meanwhile, expected funding ratio could be reduced. While when slightly changing the put cost, we would find something different:

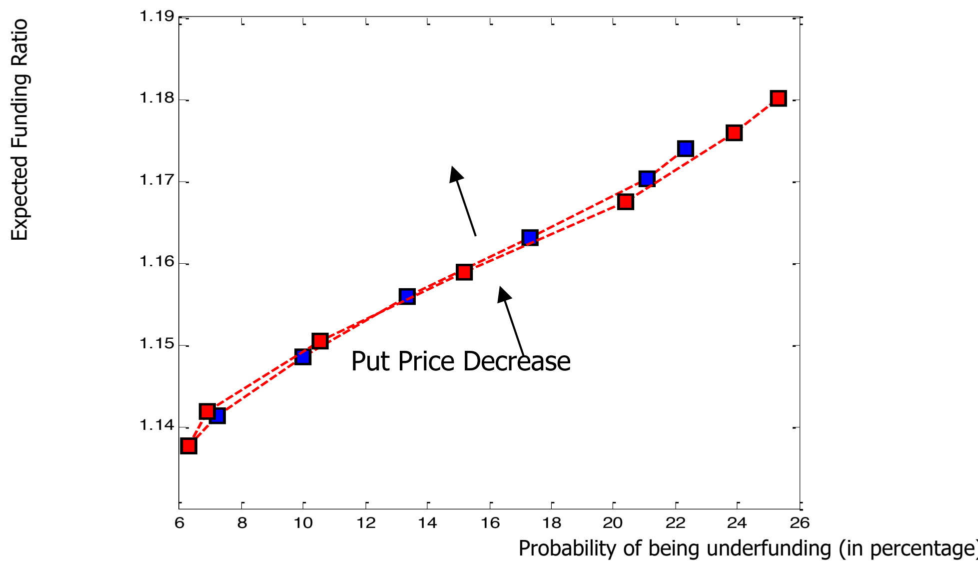


Figure 4.2.3: Comparison between non-derivatives and put-involved funds, with put price changes (initial ratio 110%)

Above is the same strategy with different option price: 3.8, which could be derived by the same risk-free rate, strike and spot price and time to maturity, while the only difference is VIX. With changing VIX to 15.23% or changing risk-free rate to 2% while keep other variables constant, we could get put price 3.8 and the blue pots in the graph, which can be considered as a break-even points. That means, after this break-even point, when put price above 3.8, the put-only strategy will reduce both risk and expected funding ratios. We get conclusion from Black Scholes Model that option price is highly sensitive to VIX; from this picture we find that put premium could influence put strategy's performance. Below we do the same test for pensions with initial funding ratio 100% and 120%:

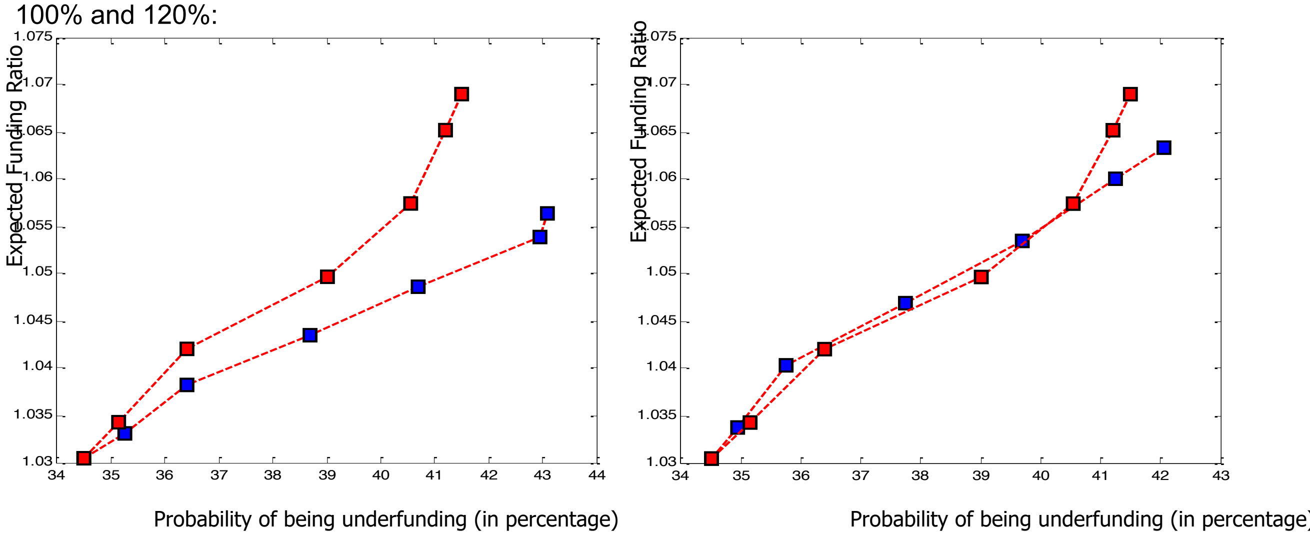


Figure 4.2.4: Comparison between non-derivatives and put-involved funds, with put option changes (Initial funding ratio 100%)

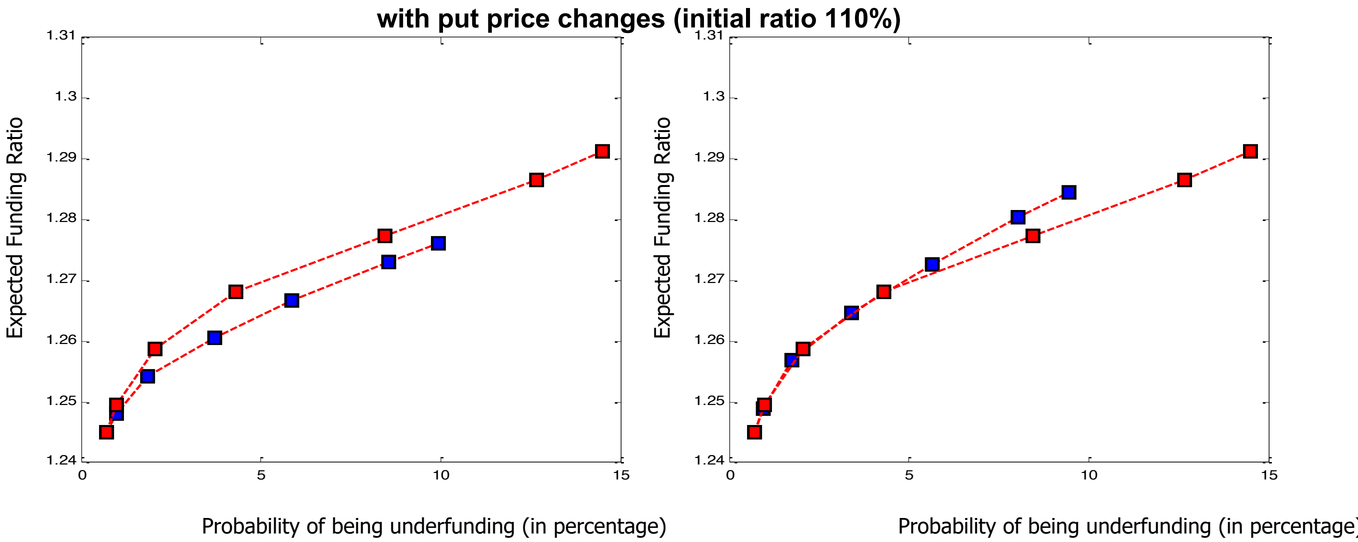


Figure 4.2.5: Comparison between non-derivatives and put-involved funds, with put option changes (Initial funding ratio 120%)

Break-even put price for both 100% and 120% is also 3.8.

Break-even: At the break even line, put involved pension experiences the same *Expected funding ratio/Risk* with non-derivatives involved pension, which could be seems as a utility equilibrium. While comparing each pair of points representing same amount of equity, we could find put strategy reduce risk and expected funding ratio.

b. Put (Out of Money)

Out of money put has been considered since it has lower premium compare to at the money put. It could be seen from below graph that out of money put (yellow line in the graph) performs even better than the at the money put. That's because of the lower premium paid for out of money put. With the same four variables (VIX 17%, spot price at 100, time 1 year and annualized risk free rate 5%) while different strike price 98, the put price derived from Black Scholes is 3.70, which is 0.74 lower than at the money put price.

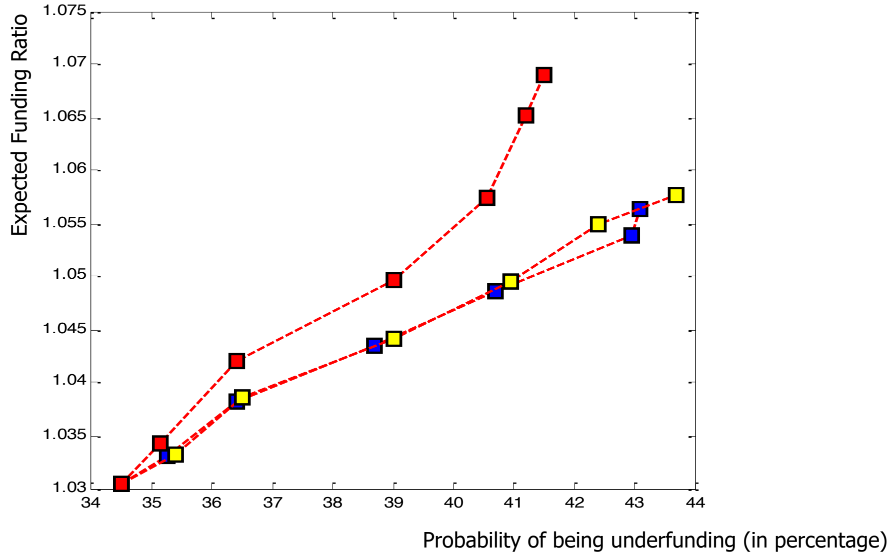


Figure 4.2.6: Comparison between non-derivatives , ATM put-involved and OTM put-involved funds , with put option changes (Initial funding ratio 100%)

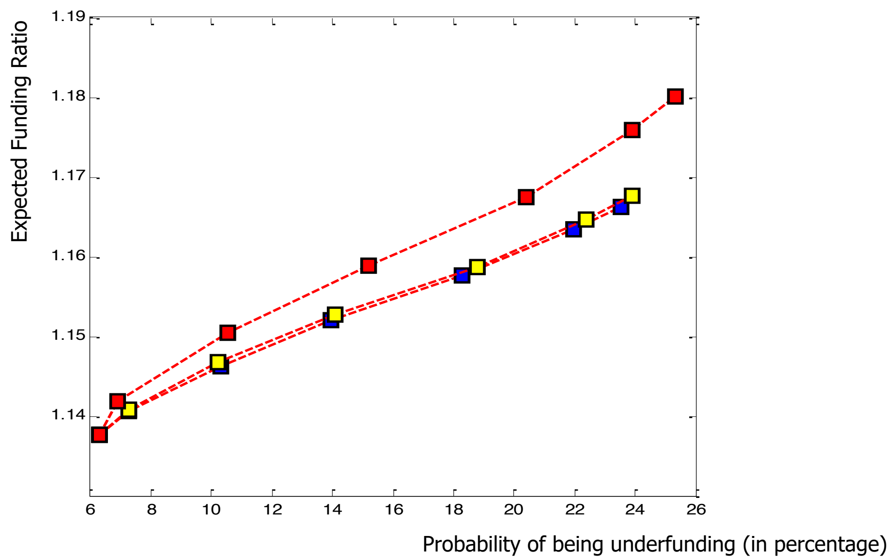


Figure 4.2.7: Comparison between non-derivatives , ATM put-involved and OTM put-involved funds , with put option changes (Initial funding ratio 110%)

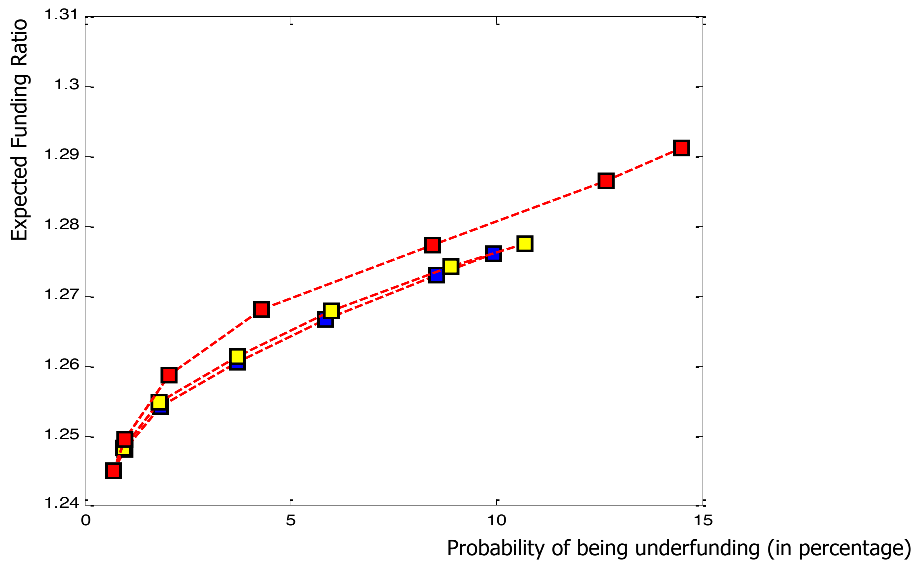


Figure 4.2.8: Comparison between non-derivatives , ATM put-involved and OTM put-involved funds , with put option changes (Initial funding ratio 120%)

Same as at the money option, out of the money option also have break-even put price. With this price out of money option could improve expected funding ratio while reduce risk of underfunding, above this price could reduce both expected funding ratio and underfunding risk.

From the graphs we find that the OTM put gives higher expected funding ratios and higher risks compare to ATM put strategy. It could be an alternative to ATM put strategy that depends on the specific pension utility function. As is shown in graph 4.2.4, if a pension is risk averse, it will prefer at the money option rather than out of the money option since ATM put strategy brings less risk compare to OTM put strategy.

c. Collar (Zero cost Put–Call Combination)

Finally, a put-call collar has been constructed by long put and short call at the same price, in which case a zero-cost protection strategy is built at the cost of potential increase of equity value. The graph given below is telling expected funding ratio and probability of under-funding, with start funding ratio 100% and above mentioned assets portfolio construction.

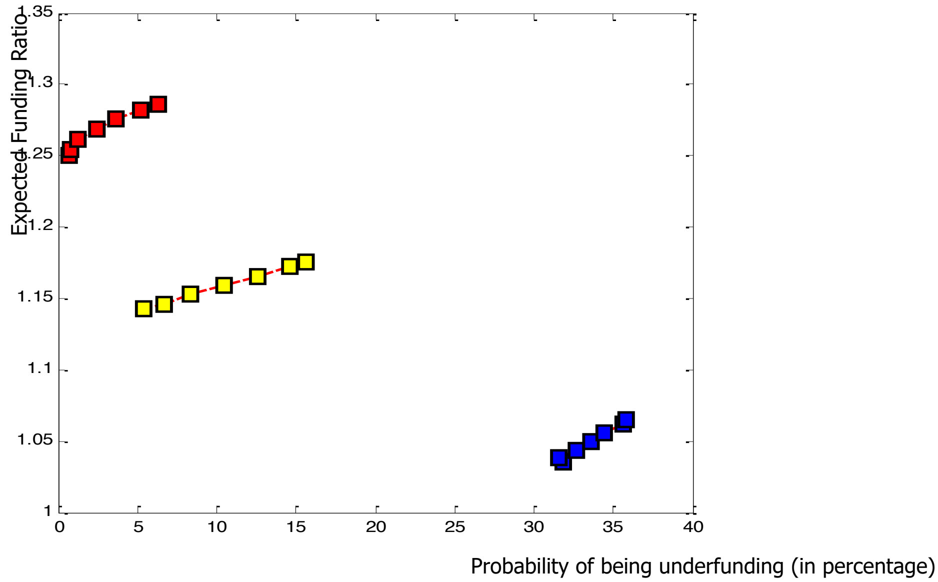


Figure 4.2.9: Expected funding profiles for funds with three initial funding levels (Collar)

Red line represents the Expected performance with no derivatives; blue line refers to 100% protection by at the money put option; Yellow line represents 100% out of money put, green line demonstrates risk-return profile of pension fund protected by collar.

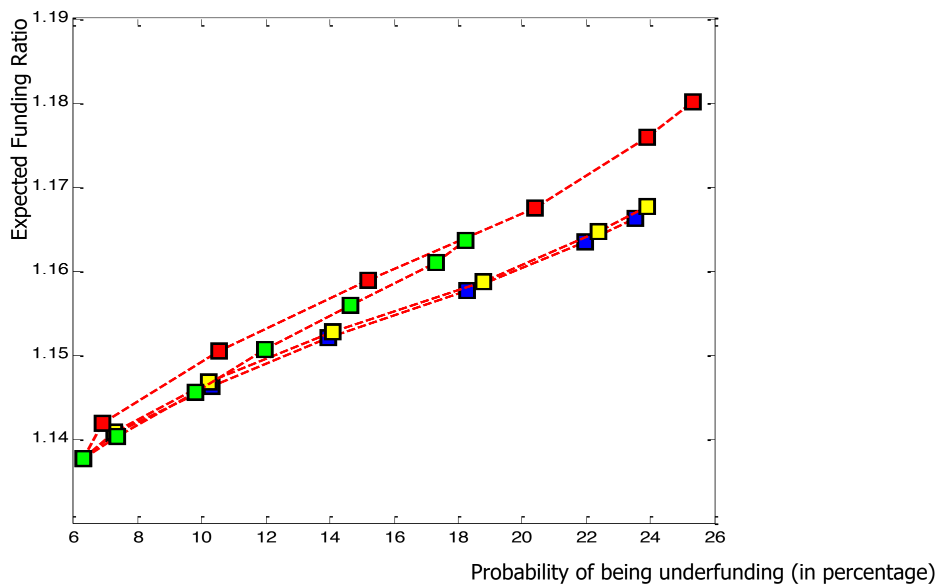


Figure 4.2.10: Comparison between non-derivatives , ATM put-involved, OTM put-involved funds and Collar-involved funds (Initial funding ratio 110%)

Different from at the money and out of money option, collar has no option premium paying problem. Theoretically collar strategy always leads to lower risk and lower expected funding ratios for pension compare to non-derivatives involved case.

Conclusion: Derivatives for Equity Implementation and Valuation

Thus, for put-only strategy, the decisive factor whether the put strategies would performs better as improving risk-return profile is the put price. More specified, the keys are risk-free rate and VIX.

In conclusion, for pensions with 100% initial funding ratio, the put strategy wouldn't perform better than non-derivatives involved case, probability of underfunding doesn't reduce because of the premium paid for put option.

For pensions with 110% initial funding ratio or above, the put strategy would reduce the probability of underfunding. Whether expected funding ratio will be higher or lower than non-derivatives case depends on the option premium. As is shown in this paper, when option premium is below certain level, including put would reduce risk of being underfunding and improve the expected funding ratio.

Collar, with its natural advantage of zero-cost, are in favored of since it could theoretically improve risk-return profile with no cost. From the investigation above we could draw the conclusion that collar is a sound strategy for improving risk-return profile of pension fund.

Collar strategy performs best among the tested strategies in the aspect of risk reduction. It reduces risk and expected funding ratios for all the pensions with 100% initial ratio, 110% initial ratio and above.

The disadvantage of using collar is the scarification of upside profit. If the equity market increases tremendously, pension with collar will experience less profit.

4.3 Derivatives for Interest Rate risk: Implementation and Evaluation

Interest Rate Swap

Swap could transform the nature of Assets & Liability profile of pension fund as was demonstrated in Chapter 3. Searching for the optimal derivatives strategy will start with hedging interest rate risk by entering a fixed receiver swap. In this case if the future interest rate goes up, fixed receiver lose money from the swap while in the liability side, the discounted liability amount also decreased; if the future interest rate goes down, pension fund benefits from paying less floating leg while suffers from an increased liability. Fixed receiver swap, in this manner, could be used as a balance adjuster as it causes assets value shift in the same direction with liability. If durations are well matched, swap could be an efficient tool for keeping stable funding ratio which is got by Assets/Liabilities.

Let's look at the pension balance sheet again:

Balance Sheet

Fixed income	900	**Liabilities**	1800
Real Estate	300	**Surplus**	200
Equity	800		
Total	2000		2000

Funding Ratio = 111%

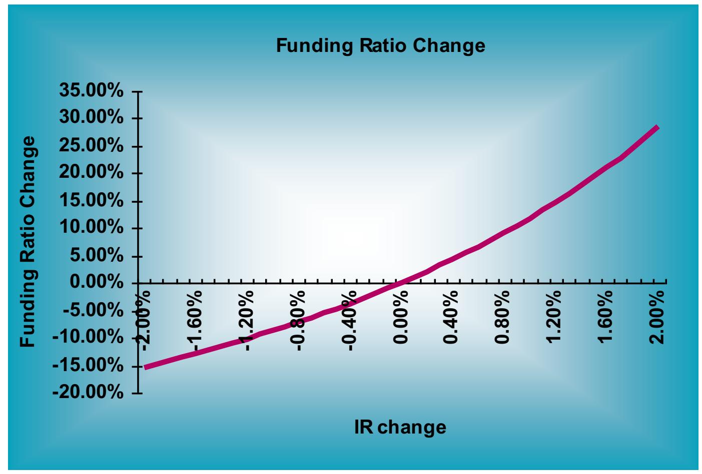
Table 4.3.1: Balance Sheet

Among the components in the balance sheet, both fixed income and liabilities (discounted future cash outflow) have interest rate sensitivities. Usually duration for fixed income is 5-7 years for a typical pension fund while for liabilities is 15 years. The pension fund suffers from interest rate risk because of the duration gap. An ideal swap strategy is entering a swap contract which fully or partially bridges the duration gap, and hence match the interest rate sensitivities between assets and liabilities. Here is an example explaining how including swap would help improve pension risk-premium profile:

First we look at pension with no derivatives involved.

Allocation:	
Funding Ratio	100%
Asset Allocation:	
Equity	0%
Fixed Income	100%
Duration:	
Fixed Income	5
Liability	15

No Derivatives involved case

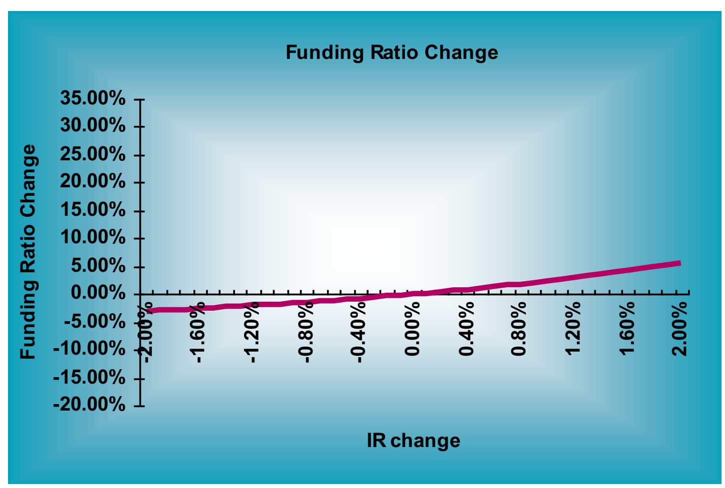


Balance:			
Assets		Liability	
Equity	0	Surplus	0
Bonds	100	Liability	100
	100		100

Then we add swap in:

Allocation:	
Funding Ratio	100%
Asset Allocation:	
Equity	0%
Fixed Income	100%
Duration:	
Fixed Income	5
Liability	15

Swap involved case



Balance:			
Assets		Liability	
Equity	0	Surplus	0
Bonds	100	Liability	100
Swap			
	100		100

Swap	
Notional	100
Duration	8

As is shown above, including swap could protect fund from interest rate decreasing. Optimized swap strategy matches the assets and Liabilities durations. A swap with duration similar to liability would be an optimal choice since then it gets same influence from yield curve shift as liability gets. Interest Rate Sensitivity of Assets:

*Value Change of Assets = - Duration of Bonds * Yield change * Bond Value*
*- Duration of Swap * Yield change * Swap Notional*
*Value Change of Liabilities = - Duration of Liabilities * IR change * Liabilities Value*

Apparently, swap with duration 8 years, notional 100 and swap with duration 5 years, notional 160 both have same value of **Duration *Swap Notional** 800, while we still need to decide which Yield change we pick, the duration of swap we picked does make difference to the interest rate sensitivities of Assets. Theoretically, the best strategy is picking some liability-matched duration for swap.

Swap: Implementation and Valuation

The optimized principals of swap are decided in such a manner that bridge the gap between (bonds duration * bonds value) and (Liability * Liability duration). It could be applied to specific pension fund in tailored fashion. In our case here, bonds have duration 7 years and liability has duration 15 years, a *x*% investment in bonds lead to a demand for swap:

Principal of swap = (Amount of Liability*15-*x*% *Amount of Assets*7)/Swap duration

By this way, the principal of swap changes for different combinations of equity and bonds assets, fully match the duration gap between assets and liability as is shown in formula above.

Let's first look at a "same yield curve" situation. Keep the yield curve unchanged for one year, only add the volatility noise. Red line represents the non-derivatives situations while blue lines represent swap involved strategy. Figure 4.3.1 shows the comparisons for initial funding ratio 110% and 120%.

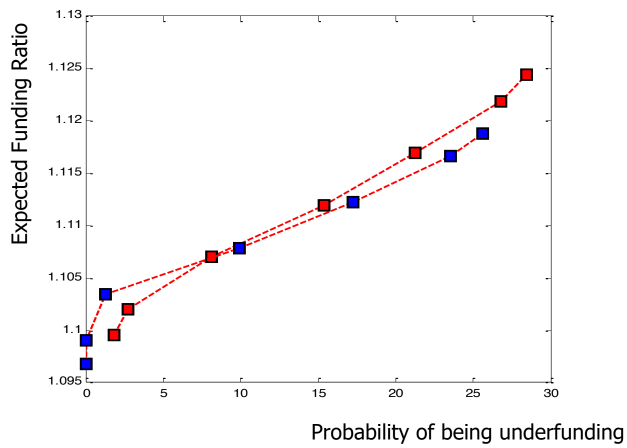


Figure 4.3.1: Comparison between a non-derivatives fund and swap-involved fund with unchanged yield curve (initial ratio 110%)

Then we back to a simulated environment (yield curve changed).

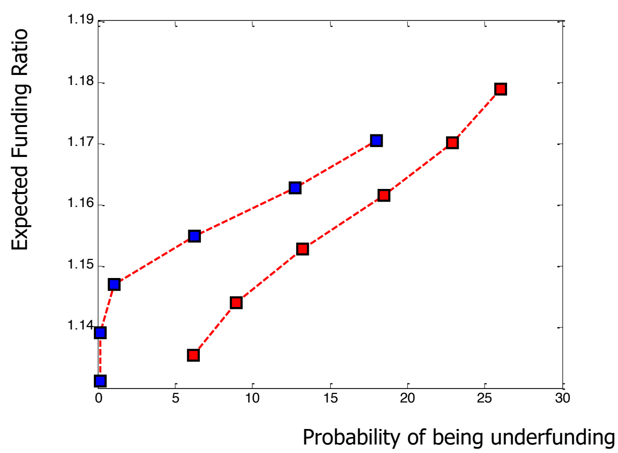


Figure 4.3.2: Comparison between non-derivatives funds and swap-involved funds with changed yield curve (initial ratio 110%)

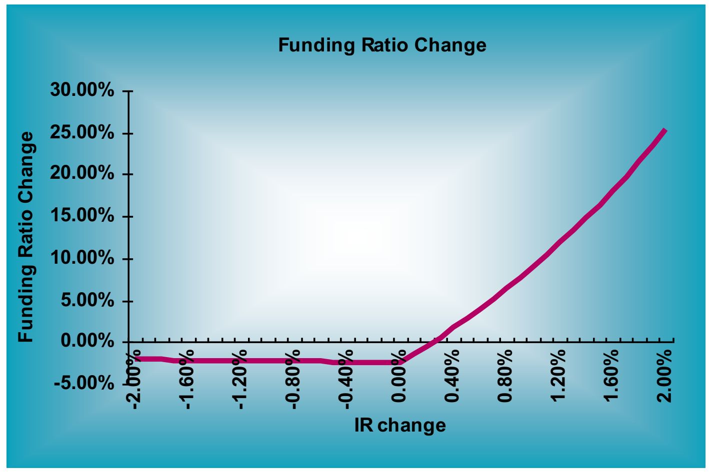
Figures 4.3.2 show how the introduction of swap changes the risk return premium of the pension funding ratio and probability of being underfunded. Notice that the simulation is conducted in a comparatively historical low interest rate (and yield curve) environment. From the graph we find that including swap in assets portfolio could reduce risk of being underfunded and lower expected funding ratio.

Swaption

Swaption gives the rights to enter a swap contract in a predetermined fixed swap rate in a certain time. As is showed in the right graph, if interest rate decrease pension fund exercise the swaption as receiving predetermined fixed swap rate, the maximum loss pension fund would experience is premium paid for swaption, in this manner assets portfolio can be kept away from down side interest rate risk.

Allocation:	
Funding Ratio	100%
Asset Allocation:	
Equity	0%
Fixed Income	97.5%
Duration:	
Fixed Income	5
Liability	15

Swaption	
Notional	100
Duration	10
Premium	2.50%



Here swaption is firstly priced by Black Model demonstrated in Chapter 2. The underlying is a swap that would start one year from now. The time to maturity for swaption is one year and the nature of underlying swap is the same as swap that has been demonstrated before (swap strategy part). The premium of swaption is paid from the assets part first, and then the rest is decided with what percentage it will be allocated to bonds and equity. The Figure 4.3.3 shows situation starting with 110%. The red line demonstrates the risk-return profile of pension fund with no derivatives and blue lines tells about the situation including swaption. It is observed from the graphs that including swaption does improving pension performance by reducing risk and expected funding ratios. This conclusion could be drawn from comparing each pair of the points with same equity assets.

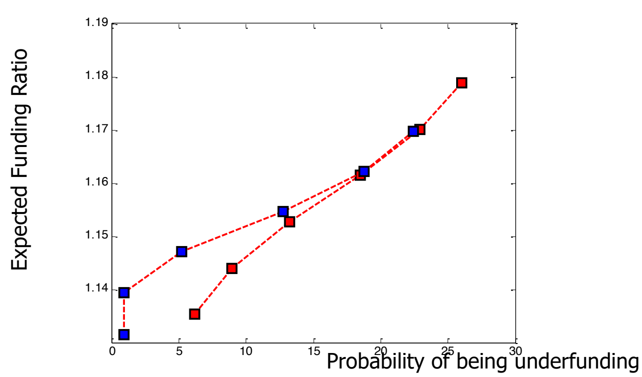


Figure 4.3.3: Comparison between non-derivatives funds and swaption-involved funds: (initial ratio 110%)

As shown in the graph, swaption can help pension reduce risk of being underfunded, meanwhile, it reduces expected funding ratios.

It should be paid attention that, with the percentage increase of equity assets, equity market risk dominates the pension risk. Swaption, which is an interest rate risk-hedging tool, gradually loses its influence to total pension risk since it only offsets the change from liability part.

Conclusion: Swap and swaption strategies

Swap and swaption have similar influences to pension funds as they both have risk reduction and lower expected funding ratio compare to non-derivatives pension with same construction. Compare these three situations: Swap including, Swaption including and non-derivatives pension:

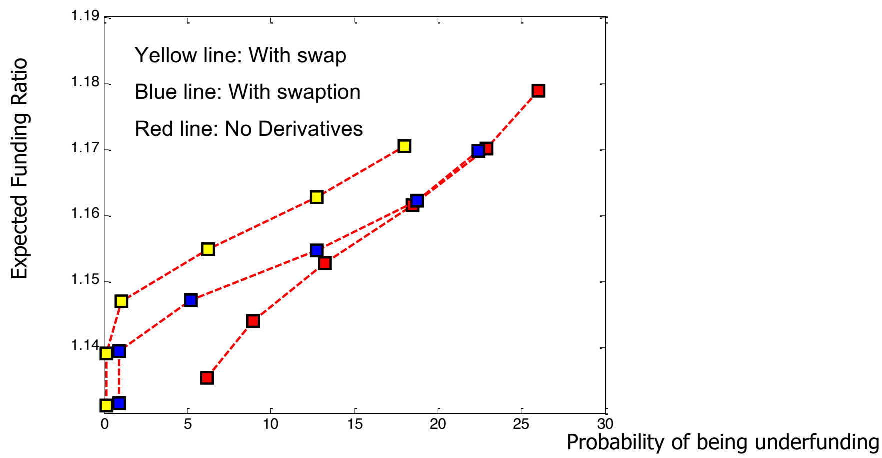


Figure 4.3.3: Comparison between non-derivatives funds, swap-involved and swaption-involved funds: (initial ratio 110%)

The swaption here is with the same underlying swap as in the swap strategy. We could find from the graph that swap-only and swaption-only strategies could both improving risk-return profile. Compare to swap-only, swaption-only has more risks of being underfunded (when equity part above 10%), and bring higher expected funding ratios. Which strategy is in favor, in this case, depends on the utility function of the pension operators. Risk averse pension manager prefer swap strategy could optimally reduce underfunded risk, while risk neutral or risk loving pension manager will consider swaption and non-derivatives strategies.

Theoretically, in an interest rate environment where interest rate is far lower than the historical average, swaption is in favored of since it could avoid loss from swap. It is demonstrated Engel, Kat and Kocken 2005, much of the interest rate risk faced by a pension fund can be eliminated by the proper use of swaps. When interest rates are well below their long-term mean, however, the risk premium on swap will be negative in the short term, which makes swaps less efficient as a hedging tool. It is in coincidence with our conclusions: in the low interest rate environment, swap helps to reduce interest rate risk while cause a lower expected funding ratio.

Engel, Kat and Kocken 2005 also gave the conclusion that swaption in preferred in a low interest rate case to avoid loss on swap, and swaption could also help reduce risk of being underfunded.

5 General Profile model

5.1 State movements

The Status of the beneficiaries are list as follows:

A: Active participants (employees)

I: Inactive participants (retirees)

D: Dead

The number of active members accounts for the cash inflows each year while the number of inactive members accounts for the cash outflows. We model the active and inactive parts separately as two homogeneous stochastic transition processes (See Appendix 5). State movements of active participants influence the collection of new contributions as assets, and the increase of liabilities. State movements of inactive participants cause decrease of liabilities.

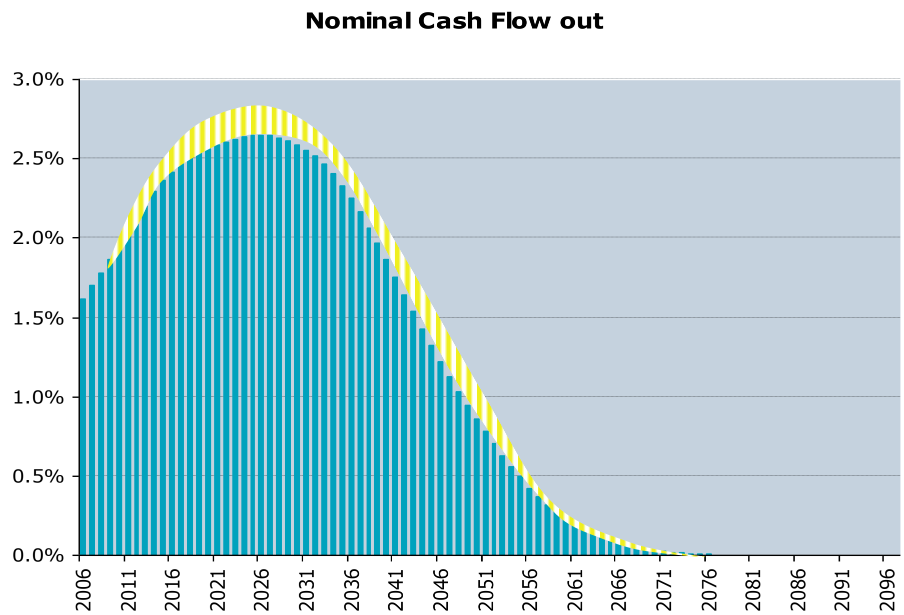


Figure 5.1: The increased liabilities are results of the number increase in active participants, and increased defined benefits.

To figure out the state movements of active and inactive participants, which influence the future cash outflows, we model the movements as Markov processes, which are used to expect the future amounts of each type of participants. Liabilities can be structured when the future numbers of participants in each state are estimated. Alternatively, we can also simply make assumptions about a fixed changing rate of liabilites, say, all the cash outflows will increase by 3% each year, consider the new participants and increased defined benefits.

5.2 Asset and Liability Evolution Process

Collecting Contributions

Contributions collected at the beginning of each year fund assets portfolio. The amount of contributions are decided by funding status, new liabilities and pension funds and life insurances' policies. Defined benefits, which will be distributed to plan participants after their retirements, keep increasing with participants' working ages until they retire. These increases, together with the amount of defined benefits for new-entering participants, form the new liabilities part. Once the amount of new liabilities has been decided, according to funds' policies, the contribution levels are decided by funding status. In this model, I made the policy assumption that if the funding ratio is higher than 120%, then the amount of new contributions are equal to that of the new liabilities; if the funding ratio is lower than 105%, then the amount of new contributions are equal to 1.5 times of that of the new liabilities; if between, then linearly related; if below 105%, then the amount to contributions is required to feed the asset value back to 105% of the liability value.

Inflation Indexation

Meanwhile, the liabilities (new-entering participants' parts exclusive) need to inflation indexations. I made the assumptions that if funding ratios are equal to or above 120%, the liabilities have full indexations of the inflation; if below 105%, no indexation; if between, linear indexations.

Portfolio Structures

During the year I assume that only asset portfolios evolve. I made the assumptions that assets only allocated to bond portfolios and equity portfolios; when adding derivatives, of course, derivatives are also in the asset portfolios. Bonds portfolios are structured with duration of 7 years, which is representative in funds; equity portfolios are assumed to be a full-diversified portfolio with the equity market return. In this model, bond portfolios and equity portfolios are 100%/0%, 80%/20%, 60&/40%, 40%/60%, 20%/80, 0%/100% included.

When adding derivatives, relevant derivatives are firstly priced. Swaptions are priced by Black formula, as described in Black, Fisher 1976. All the swaptions are assumed to be at the money swaptions. When decide the assets allocations, first buy derivatives, the rests are allocated to bond portfolios and equity portfolios. In individual swap and swaption strategies, the amounts of notional are decided in such a way that the interest rate risks are fully hedged, that is, the duration gaps between asset portfolios and liability portfolios are covered. When use dynamic swaption and swap strategies, however, I made a simple assumption to invest in swaptions when interest rates below 5% and switch to swap if it is above 5%; this assumption is too simple, when considering what is a low interest rate environment to use swaption, it is hard to make a clear definition.

Then asset values after one year can be calculated with the VAR model, which provides simulated yield curve and equity premium results.

Defined Benefits: Cash Outflow at the end of the year

At end of the year, subtract defined benefits from both asset portfolios and liability portfolios, the funding status are decided. Then next year starts with new contributions collection and inflation indexations like demonstrated before.

6 Long Term Analysis

6.1 non-derivatives involved case

Based on the short-term analysis, selected strategies are applied for long-term analysis, which has 5, 10, or 25 years' or longer horizon. When strategies are used to an extended period, option, swap and swaptions strategies will be reconstructed in a rolling manner, which will be explained in the following chapters.

Based on the generated economic scenarios (Chapter 3: VAR) and estimated pension fund's general profile (Chapter 5), we first simulate the risk-return profiles for non-derivatives involved pension fund on 5 year, 10 year and 25 year horizon. Below are the tables:

		100% Bonds	80% Bonds	60% Bonds	40% Bonds	20% Bonds	0% Bonds
5 year							
	Expected Funding Ratio	1.14	1.18	1.24	1.30	1.37	1.45
	Probability of Underfunding	6.5%	7.7%	10.6%	13.5%	15.8%	17.4%
	Inflation Indexation	55.5%	63.0%	66.9%	69.2%	70.5%	71.3%
	New Contribution/ New Liability	1.42	1.44	1.54	1.69	1.88	2.08
10 year		100% Bonds	80% Bonds	60% Bonds	40% Bonds	20% Bonds	0% Bonds
	Expected Funding Ratio	1.14	1.19	1.28	1.40	1.54	1.69
	Probability of Underfunding	6.9%	8.0%	9.3%	10.6%	11.7%	12.5%
	Inflation Indexation	52.7%	63.7%	71.3%	74.7%	76.4%	77.3%
	New Contribution/ New Liability	1.45	1.44	1.49	1.59	1.70	1.82
25 year		100% Bonds	80% Bonds	60% Bonds	40% Bonds	20% Bonds	0% Bonds
	Expected Funding Ratio	1.13	1.19	1.34	1.58	1.89	2.28
	Probability of Underfunding	6.1%	7.8%	7.7%	8.0%	8.3%	8.8%
	Inflation Indexation	50.3%	62.9%	72.7%	79.5%	82.5%	83.9%
	New Contribution/ New Liability	1.44	1.41	1.40	1.42	1.49	1.57

Table 6.1: Expected profiles for non-derivatives funds

From the table we find that, the differences in probabilities of underfunded between 100% bonds assets and 100% equity assets become smaller with time going.

From the table we also find that 100% investment in Bonds brings stable expected funding ratios and expected probabilities of being underfunded, while 100% investment in equity leads to higher expected funding ratios and meanwhile, probabilities of underfunded gradually decrease. One possible explanation is that on average equity assets bring more risk return, which helps pension fund reach a high funding ratio, reduces the risk of being underfunded next year.

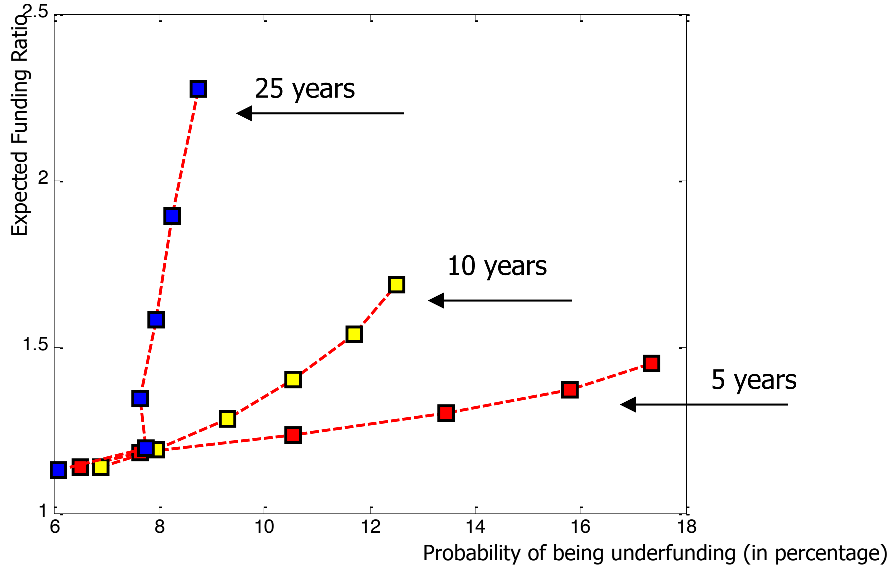


Figure 6.1: Expected profiles for non-derivatives funds

6.2 Equity Derivatives

Put strategy works in a rolling manner, puts protecting all the equity assets are bought at the start of the year with a time to maturity of 1 year. After one year put option will be exercised is the pay-off is positive, otherwise it expire with no value.

6.2.1 At The Money Put

First we check the Put-only strategy's performance in long term. Let's check the comparisons in Figure:

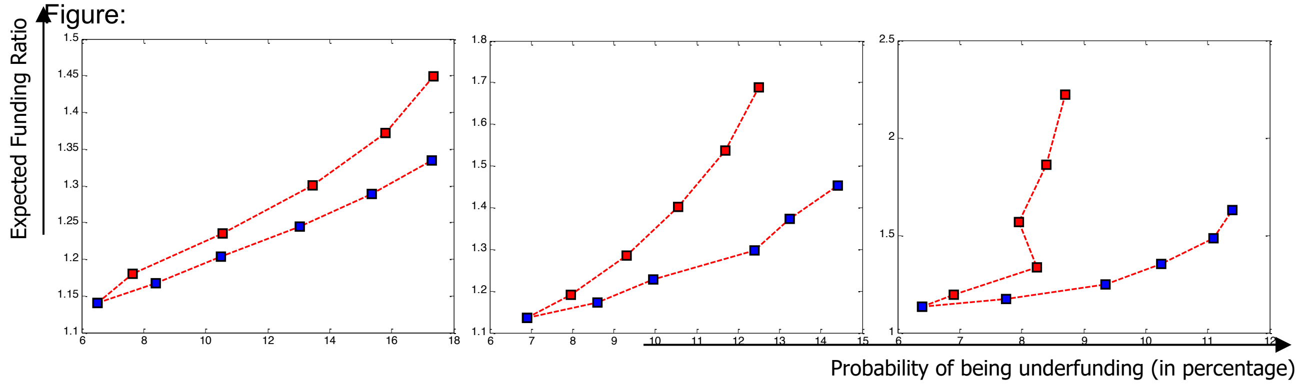


Figure 6.2.1: Comparisons of expected funding status between Non-derivatives funds and Put-only involved funds on 5, 10, 25 years horizons

Apparently, put-involved pension fund will get lower expected funding ratios and higher probabilities of being underfunded compare to corresponding non-derivatives pension funds (see table in chapter 6.1). In other words, put-only strategy cannot improve, or even harm pension fund risk return profile in long term.

6.2.2 Collar

The short term well-performed collar strategy is also being test in long horizons. With no doubt it reduces pension fund's risk of being underfunded in short run, however, collar strategy brings no advantage in 10 years horizon. For 25 years time horizon, as is in the 10 years, collar increase the risk of being underfunded. One explanation is that including collar in assets portfolio cut the potential upside return as well as avoiding the downside risk. As a result, in long run, collar makes pension suffer more loss of upside return by an upper boundary than benefit got from protecting by a lower boundary.

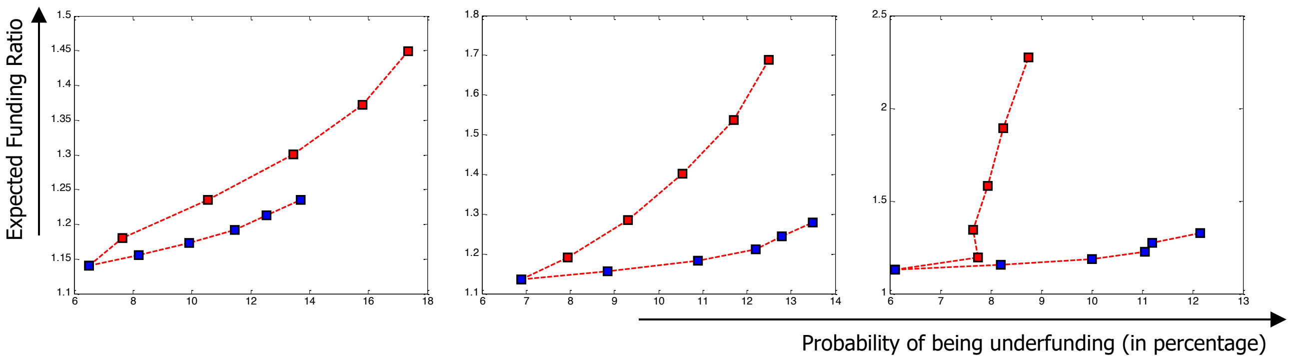


Figure 6.2.2: Comparisons of expected funding status between Non-derivatives funds and Collar involved funds on 5, 10, 25 years horizons

Collar could be a helpful tool to improve pension funds' risk return profile in short term. However, careful test and investigation should be done before applied collar strategy, since its performance is highly correlated to time horizon and the structure of assets portfolio.

6.2.3: Inflation Indexations

How much inflation indexations participants can get, and what contribution levels participants should pay, are of great importance. We get the following figures on 5, 10 and 25 years' horizon.

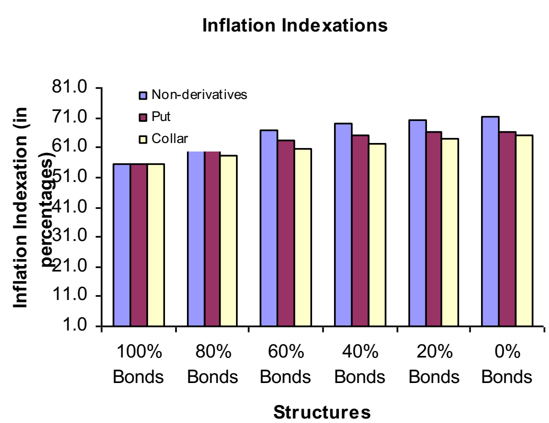


Figure 6.2.3a: Comparisons of expected Inflation Indexations among Non-derivatives funds, Put involved funds and Collar involved funds on 5 years horizons

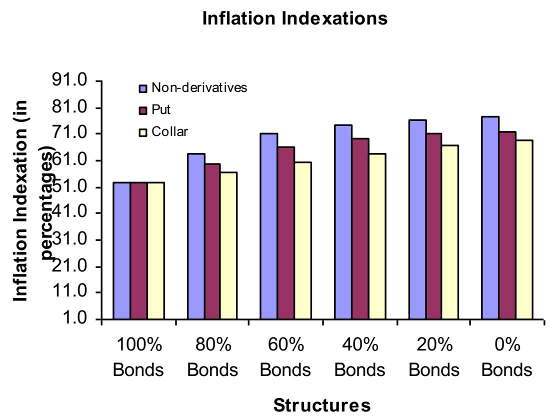


Figure 6.2.3b: Comparisons of expected Inflation Indexations among Non-derivatives funds, Put involved funds and Collar involved funds on 10 years horizons

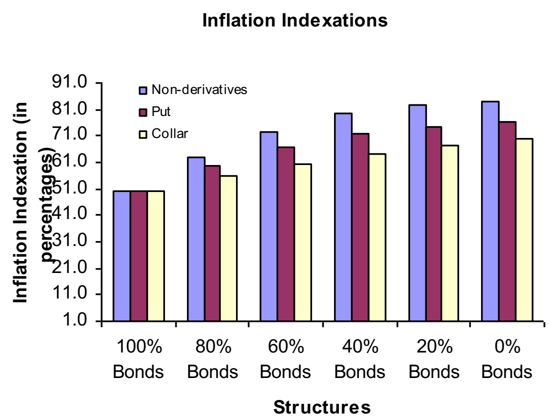


Figure 6.2.3c: Comparisons of expected Inflation Indexations among Non-derivatives funds, Put involved funds and Collar involved funds on 25 years horizons

From the figures above we find that inflation indexations are negatively influenced by put and collar strategies on all the three time horizons. Besides, compare to put, the collar strategy brings even lower inflation indexations, which caused by the low expected funding ratios because of the collar strategy.

6.2.4: Contribution Levels

Contribution levels are essential factors for pension funds' health. Participants and pension managers expect low contribution levels, which represent the cost of their pensions.

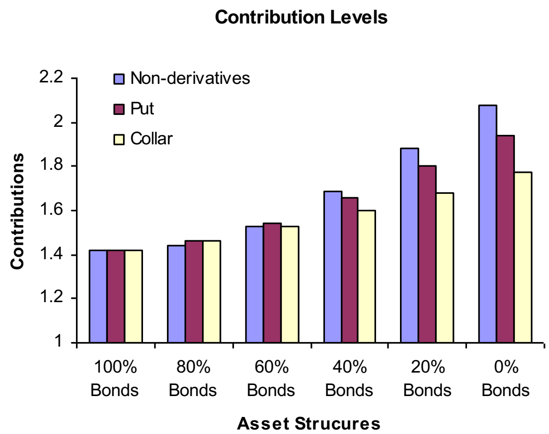


Figure 6.2.4a: Comparisons of expected contribution levels among Non-derivatives funds, Put involved funds and Collar involved funds on 5 years horizons

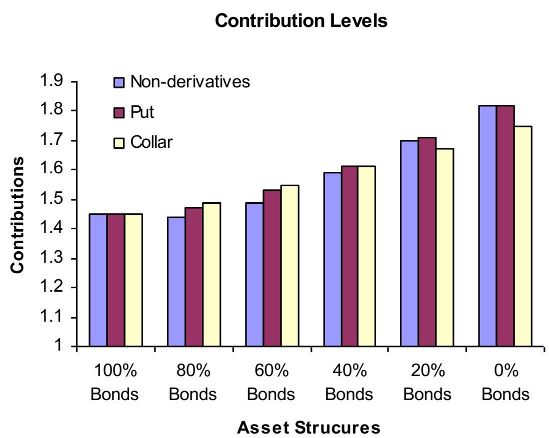


Figure 6.2.4b: Comparisons of expected contribution levels among Non-derivatives funds, Put involved funds and Collar involved funds on 10 years horizons

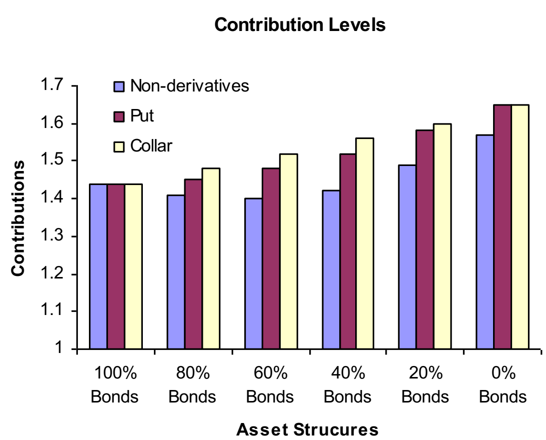


Figure 6.2.4c: Comparisons of expected contribution levels among Non-derivatives funds, Put involved funds and Collar involved funds on 25 years horizons

From the figures we find in short run, for instance, 5 years, contributions are lower because of the equity derivatives, however, in medium term and long term, derivatives bring higher contribution levels.

6.3 Interest Rate Derivatives

Interest rate derivatives strategies work in a rolling manner. All swaps and swaptions are valued at marked-to market base. Assumptions are made that the swap and swaption market are liquid enough that at the start of each year, pension managers can freely adjust their swap and swaption positions to make ideal asset portfolio, with no transaction cost. These assumptions are only theoretically held.

We want to know if there are differences of strategy performances in different interest rate environments. For this reason, all the interest rate risk hedging strategy' evaluations are conducted in two environments: One is a smooth interest rate environment, in which current interest rates are close to historical mean; another is an increasing interest rate environment, in which current interest rate are lower than historical mean, while expected to bounce to the historical mean level.

Let's first have a look at a relatively short time horizon: 5 years' horizon. Look at the funding ratios on 5 years' horizon. Left figure is funding status results of a smooth environment while right is of an increasing environment.

Red pots refer to the non-derivatives; yellow pots refer to the swap-included cases with the same corresponding allocation; blue pots are swaption cases and greens are dynamic swap and swaption cases.

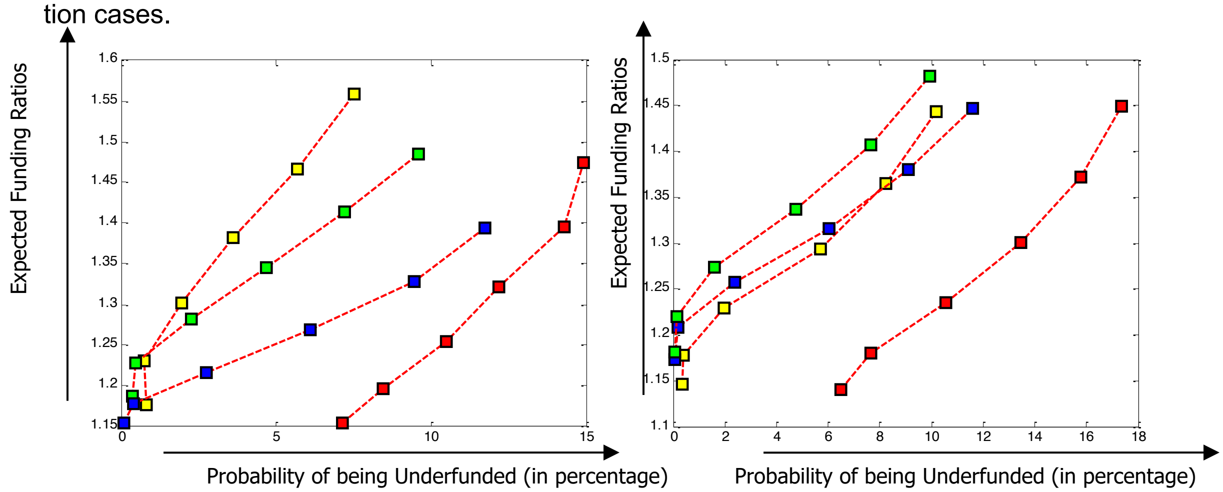


Figure 6.3.1: Funding status after 5 years

In smooth interest rate environments, the swaption strategy bring the highest expected funding ratio and the dynamic swaption and swap strategy is slightly better than the other two when compare the risk reduction function (except the 100% equity case). In the right figure, the dynamic strategy reduces most risk and bring the highest expected funding ratios; the swaption strategy, comapre to the swap strategy, brings higher expected funding ratios and lower risks in most of the structured assets, however, when there are equity assets, bring less risk reduction function than the swap strategy.

Other two important evaluation factors are contribution levels and inflation indexations. In the ideal pension plan, participants pay low contributions and receive high inflation indexations. Contributions are regarded as pension cost, which implies how much contributions needed to be paid with one unit increase in liabilities. On a 5 years' horizon, including three different strategies will lead to the following contribution levels and inflation indexations in two predetermined interest rate environments:

The contribution level is another key factor implying funds' health.

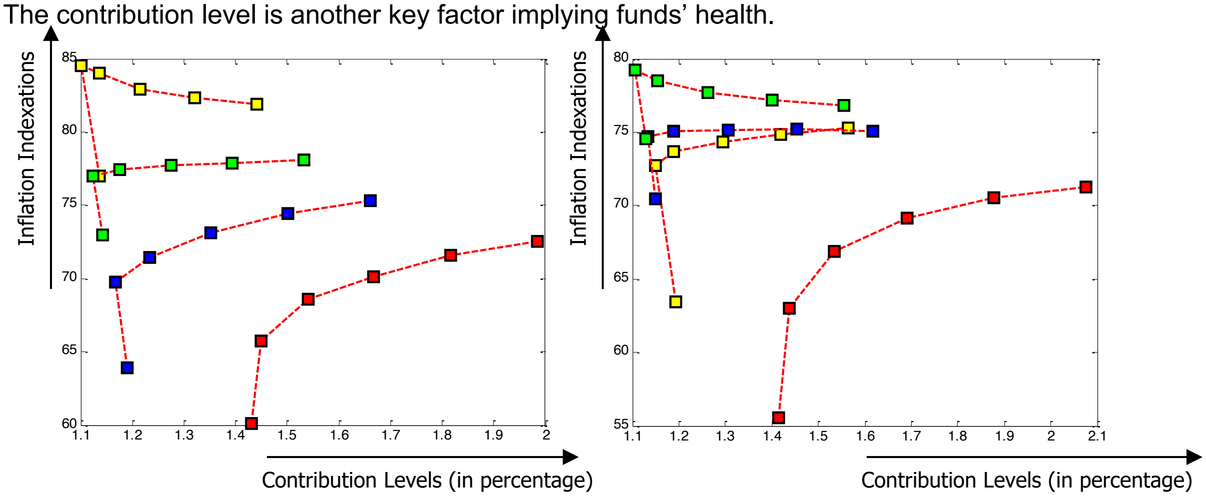


Figure 6.3.2: Inflation Indexations and Contribution levels at the beginning of 5th year

In the 5 years' horizon, all the three derivatives-involved strategies bring lower contribution levels and higher inflation indexations, which could satisfy plan participants and benefit the pension funds. In smooth interest rate environment, the swaption strategy bings the lowest contribution levels and highest inflation indezations; In the increasing interest rate environment, the dynamic swaption&swap strategy performs the best.

If look at the pension funds profiles in a longer term, say, 10 years' horizon, the advantages of using interest rate derivatives still hold.

Red pots: non-derivatives; Yellow pots: swap-included;

Blue pots: swaption; Green pots: dynamic swap and swaption.

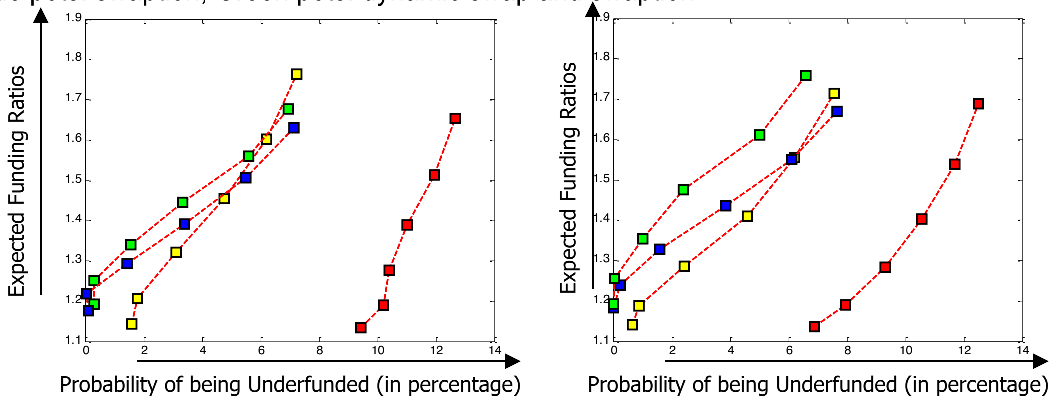


Figure 6.3.3: Funding status after 10 years

In different interest rate environments, three strategies have similar performances. Dynamic swaption and swap strategy, within every asset structures, wins. In smooth interest rate cases the swaption strategy performs better than the swap strategy. In increasing interest rate environment roughly the same, only when there is 100% equity assets, the swaption strategy is slightly worse than the swap. The contribution levels and inflation indexations are in the figures below.

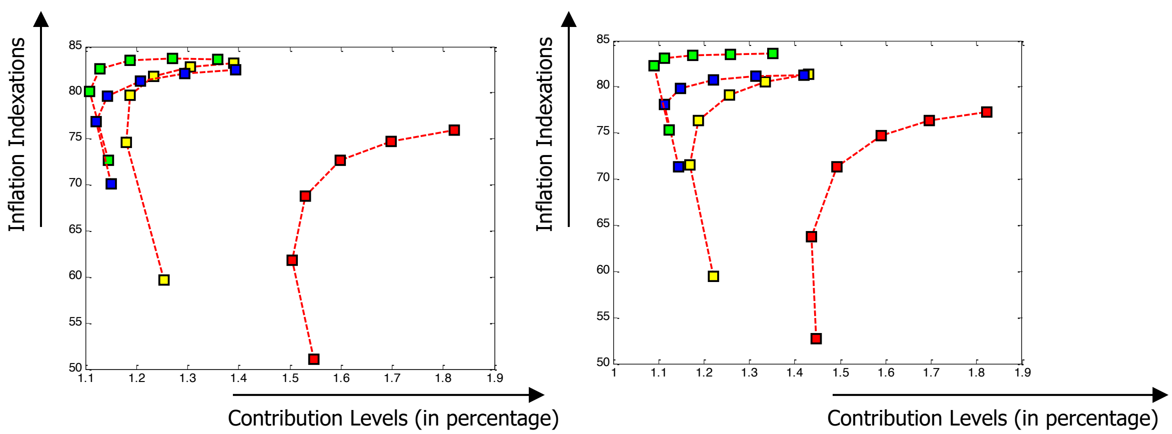


Figure 6.3.4: Inflation Indexations and Contribution levels at the beginning of 10th year

From the figures above, the dynamic swaption and swap strategy wins in any case.

In conclusion, interest rate derivatives are useful risk hedging tools. Any of these three strategies can help improve funding status and pension health. In medium and long term, a dynamic swap and swaption strategy always performs best since it is interest rate dependent.

Ⅳ Conclusions

This project aims at investigations of structuring derivatives strategies to hedge euqity risks and interest rate risks. Short term analysis and long term analysis are conducted based on a scenario method. The most important findings are listed as follows:

1, equity options could add value to ALM by reducing equity risks in assets portfolio. In coincidence with Capelleveen (2004), we also found that in short term, if well structured to fit asset portfolios, put options and collars drastically reduce equity risks.

However, in medium term and long term, say, over a 10 years' horizon, neither put options nor collars can reduce risk; Moreover, because of the high premiums paid for put options and limitations of upside potentail of collar, including put options and collars in pension funds even causes worse funding status than non-derivatives involved case.

Thereby, from a long picture, it is not suggested to use these equity derivatives. Pension fund managers need careful considerations before adding equity derivatives into their portfolios.

2, Interest rate derivatives can improve funds'performance in short term, medium term and long term. A dynamic swaption and swap strategy are the best solution because it not only reduce underfunding risks, also greatly reduce the contribution level, which befenit the pension plan participants.

Appendix

Appendix 1: Black Scholes Formula

(Black, F., M.Scholes 1973):

$$P(S,T) = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

Where

$$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

K is the strike price at maturity, S_0 is the current stock price, T is the time to maturity, r is the continuously compounded risk-free rate and σ is the implied volatility. *N(x)* is the cumulative probability distribution function for a variable that is normal distributed with a mean of zero and a standard deviation of 1.0 (i.e., it is the probability that such a variable will be less than *x*.)

Appendix 2: Black Model

L is principal, T refers to maturity date of the option, F is forward swap rate for a contract with maturity T, F0 is value of F at time zero, Rx is Strike fixed rate, P(t,T) is price at time t of a zero-coupon bond paying $1 at time T, σ is volatility of F, and m is the frequency of m times per year.

$$LA[R_X N(-d_2) - F_0 N(-d_1)]$$

where

$$d_1 = \frac{\ln(F_0 / R_X) + \sigma^2 T / 2}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln(F_0 / R_X) - \sigma^2 T / 2}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$

$$A = \frac{1}{m} \sum_{i=1}^{mn} P(0, t_i)$$

At the time of maturity, if spot fixed swap rate is higher than strike rate, pension fund won't exercise the swaption but switch to a swap since pension fund is fixed receiver.

In our implementation, $A = \frac{1}{m}\sum_{i=1}^{mn} P(0, t_i)$ could be easily derived from simulated yield curve, while is demonstrated in Chapter 3 VAR model, *m* is defined as 4 since it supposed to be quarterly swap. σ is volatility of fixed swap rate.

Appendix 3: Nelson Siegel Model

Nelson Siegel Model

Here we introduce the framework which we use for fitting and forecasting the yield curve. Well-known Nelson and Siegel (1987) curve is well-suited to our ultimate forecasting purposes. Diebold and Li (2004) showed that the three coefficients in the Nelson-Siegel curve may be interpreted as latent level, slope and curvature factors. Diebold-Li also argued that the nature of the factors and factor loadings implicit in the Nelson-Siegel model facilitate consistency with various empirical properties of the yield curve that have been catalogued over the years.

In particular, Nelson and Siegel (1987), as extended by Siegel and Nelson (1988), work with the forward rate curve,

$$f_t(\tau) = \beta_{1t} + \beta_{2t} e^{-\lambda_t \tau} + \beta_{3t} \lambda_t \tau \ e^{-\lambda_t \tau}.$$

The Nelson-Siegel forward rate curve can be viewed as a constant plus a Laguerre function, which is a polynomial times an exponential decay term and is a popular mathematical approximating function. The corresponding yield curve is

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau}\right) + \beta_{3t}\left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau} - e^{-\lambda_t \tau}\right)$$

Here is the interpretation for the parameters in the Nelson-Siegel model. The parameter λ_t governs the exponential decay rate; small values of λ_t produce slow decay and can better fit the curve at long maturities, while large values of λ_t produce fast decay and can better fit the curve at short maturities. λ_t also governs where the loading on β_{3t} achieves its maximum.

We interpret β_{1t}, β_{2t} and β_{3t} as three latent dynamic factors. The loading on β_{1t} is 1, a constant that does not decay to zero in the limit; hence it may be viewed as a long-term factor. The loading on β_{2t} is a function that starts at 1 but decays monotonically and quickly to 0; hence it may be viewed as a short-term factor. The loading on β_{3t} is, which starts at 0 (and is thus not short-term), increases, and then decays to 0 (and thus is not long-term); hence it may be viewed as a medium-term factor. We plot the three factor loadings in figure 3.2.

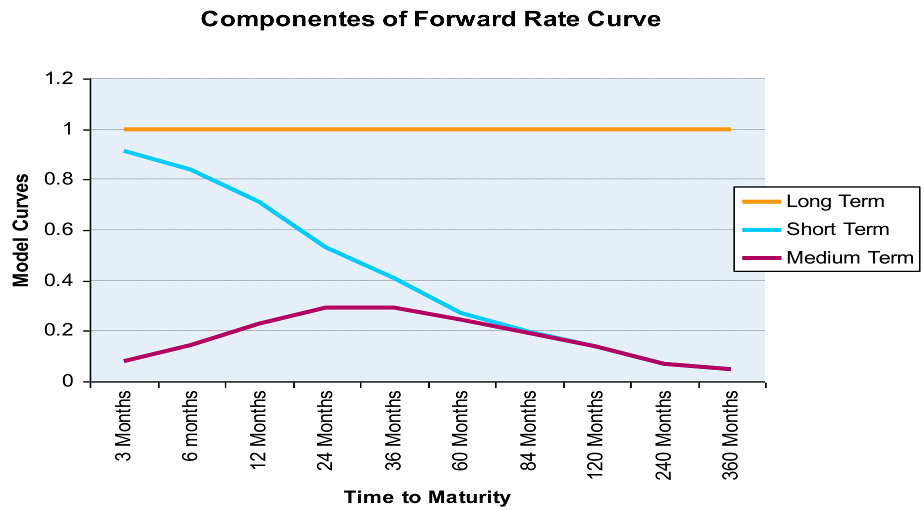


Figure 3.2: Components of Forward Rate Curve

An important insight is that the three factors, which following the literature we have thus far called long-term, short-term and medium-term, may also be interpreted in terms of level, slope and curvature as mentioned before. The long-term factor β_{1t} , for example, governs the yield curve level. In particular, one can easily verify that $y_t(\infty) = \beta_{1t}$. Alternatively, note that an increase in β_{1t} increases all yields equally, as loading is identical at all maturities, thereby changing the level of the yield curve.

We have seen that β_{1t} governs the level of the yield curve and β_{2t} governs its slope. It is interesting to note, moreover, that the instantaneous yield depends on *both* the level and slope factors, because $y_t(0) = \beta_{1t} + \beta_{2t}$.

Finally, the medium-term factor β_{3t} is closely related to the yield curve curvature, it has little effect on very short or very long yields, which load minimally on it, but will increase medium-term yields, which load more heavily on it, thereby increasing yield curve curvature.

As stated in the paper Diebold and Li (2004), we fixed λ_t at a certain value (0.0598) which maximizes the loading on β_{3t} , the medium term factor. Then β_{1t} , β_{2t} and β_{3t} are estimated based on regressions.

Appendix 4: Comparison among OLS, Yule-Walker and Stepwise

Here we go though OLS, Yule Walker and Stepwise Least Square, we found the same result that *p*=1 is the most suitable choice for order of lag. Here are the coefficients matrices Φ and Noise matrix *C* got from three methods:

OLS	Mean	β2 -MEAN	β3 -MEAN	β1 -MEAN	VIX -MEAN	Inflation -MEAN	Equity Premium -MEAN
B2-MEAN	-0.75	0.72	0.12	0.15	0.01	-0.01	0.02
B3-MEAN	-2.36	0.19	0.78	-0.17	-0.01	0.05	0.00
β1-MEAN	7.44	-0.03	0.05	0.99	-0.01	-0.05	0.00
VIX – MEAN	18.40	0.02	0.38	-0.14	0.65	-0.34	0.08
Inflation-MEAN	3.12	0.04	0.01	0.03	0.00	0.76	0.00
EquityP- MEAN	1.71	0	0	0	0	0	0

Table 3.4.5: OLS estimated coefficients

With Noise Matrix:

OLS Noise	Beta 2	Beta 3	Beta 1	VIX	Inflation	Equity Premium
Beta 2	1.30	0.00	0.26	-1.78	0.07	1.62
Beta 3	0.00	0.30	-0.14	-0.40	-0.01	1.01
Beta 1	0.26	-0.14	0.22	-0.21	0.10	-0.46
VIX	-1.78	-0.40	-0.21	26.40	-0.04	-22.25
Inflation	0.07	-0.01	0.10	-0.04	0.39	-0.44
EquityP	1.62	1.01	-0.46	-22.25	-0.44	59.76

Table 3.4.6: Noise Matrix for OLS Estimation

Yule Walker	Mean	β2 -MEAN	β3 -MEAN	β1 -MEAN	VIX -MEAN	Inflation -MEAN	EquityP -MEAN
B2-MEAN	-0.75	0.71	0.13	0.15	0.01	-0.01	0.02
B3-MEAN	-2.36	0.21	0.73	-0.17	0.00	0.12	0.00
β1-MEAN	7.44	-0.04	0.10	1.00	-0.01	-0.11	0.00
VIX – MEAN	18.40	0.01	0.42	-0.13	0.64	-0.39	0.08
Inflation-MEAN	3.12	0.03	0.04	0.03	-0.01	0.72	-0.01
EquityP – MEAN	1.71	0	0	0	0	0	0

Table 3.4.7: Yule-Walker estimated coefficients

Noise Matrix:

Yule-Walker Noise	Beta 2	Beta 3	Beta 1	VIX	Inflation	Equity Premium
Beta 2	1.30	0.00	0.26	-1.78	0.07	1.59
Beta 3	0.00	0.31	-0.15	-0.41	-0.01	1.03

Beta 1	0.26	-0.15	0.23	-0.21	0.10	-0.49
VIX	-1.78	-0.41	-0.21	26.40	-0.04	-22.25
Inflation	0.07	-0.01	0.10	-0.04	0.39	-0.44
EquityP	1.59	1.03	-0.49	-22.25	-0.44	59.76

Table 3.4.8: Noise Matrix for Yule-Walker Estimation

Table 3.4.8

Stepwise LS	Mean	β2 -MEAN	β3 -MEAN	β1 -MEAN	VIX -MEAN	Inflation -MEAN	Equity premium -MEAN
β2-MEAN	-0.75	0.75	0.08	0.06	0.02	0.02	0.03
β3-MEAN	-2.36	0.18	0.80	-0.12	-0.01	0.02	0.00
β1-MEAN	7.44	0.01	0.02	0.95	0.00	-0.02	0.00
VIX – MEAN	18.40	-0.02	0.55	0.55	0.59	-0.94	0.04
Inflation-MEAN	3.12	0.03	0.02	0.05	-0.01	0.77	0.00
EquityP– MEAN	1.71	0	0	0	0	0	0

Table 3.4.9: Stepwise estimated coefficients

Noise matrix:

Stepwise LS Noise	Beta 2	Beta 3	Beta 1	VIX	Inflation	Equity Premium
Beta 2	1.33	-0.01	0.27	-1.96	0.06	1.59
Beta 3	-0.01	0.31	-0.14	-0.32	0.00	1.01
Beta 1	0.27	-0.14	0.22	-0.28	0.09	-0.44
VIX	-1.96	-0.32	-0.28	28.09	0.02	-21.96
Inflation	0.06	0.00	0.09	0.02	0.39	-0.36
EquityP	1.59	1.01	-0.44	-21.96	-0.36	59.76

Table 3.4.10: Noise Matrix for Stepwise Estimation

As going through these three estimated results, we conclude that they give similar estimated results. As suggest by Neumaier and Schneider [2001], we finally adopt Stepwise least square, which demonstrated in the paper to be the most accurate and efficient way for estimation.

Appendix 5: State Movement Model

Active Population Dynamics

To describe the dynamics of **Pt**, the pension fund's active population, a time homogenous stochastic transition matrix Π is defined. All possible combinations of status define one specific state constitute the vector space **Z**. For illustrative purpose, a pension fund active population of employees is considered. Initially, the number of population is *M*; *M* is according to a specified distribution with mean M. At time 0, the number of population could be partitioned into one of subsets [1], [2]… [M-1], [M], [M+1]…, [2M-1], [2M], that's say, 2M state of populations; with the assumption that *M* would never below 1 or above 2M. Define Z=[1,2,…,2M] as space vector. Each subset represents a state of population. At time 1, the number of population would change or keep constant. For instance, at time 0 the number of population is in state1 [1], then at time 1, it could stay at [1], or jump to [2] or even other states. We set a matrix Π to capture the probability to migrate from state i to state j, where π_{ij} refers to the element in the *i*th row and *j*th column:

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \cdots\cdots & \pi_{1,2M} \\ \pi_{21} & \pi_{22} & \pi_{23} & \cdots\cdots & \pi_{2,2M} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & \\ \pi_{2M,1} & \pi_{2M,2} & \pi_{2M,3} & \ldots & \pi_{2M,2M} \end{bmatrix}$$

For instance, π_{12} refers to the probability jumping from state1 [1] at time 0 to state2 [2] at time 1, etc. Thus, the elements π_{ij} of Π satisfy the following conditions:

$$\pi_{ij} \geq 0, \forall i, j \in Z,$$
$$\sum_{j=Z} \pi_{ij} = 1, \forall i \in Z$$

Getting π_{ij} is the key for solve the stochastic transition process for population.

Making following further assumptions:

M: Number of population at time 0

k: Number of population at time 1

NN: Number of new comers during time 0 and time 1

L: Number of people left during time 0 and time 1

Apparently,

$$\pi_{M,k} = P(M + NN - L = k) = \sum_{n=0,L=0}^{k,M} P(M + NN - L = k \mid NN = n) \times P(NN = n)$$
$$= \sum_{n=0,L=0}^{k,M} P(L = M + n - k) \times P(NN = n) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

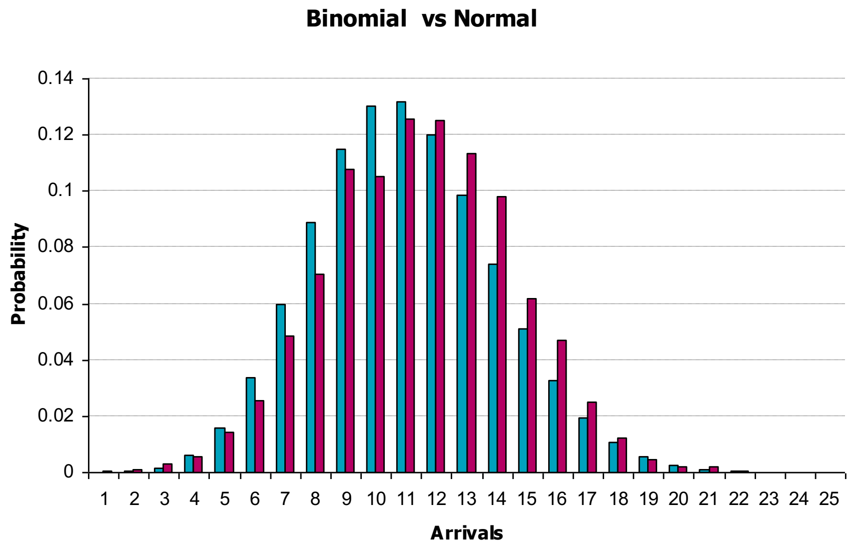
Leaving process: Binomial Process

As is showed in the formula, $P(L = M + n - k)$ represents the leaving process, which describes the probability of L people leave during time 0 and time 1. For given *M* the sum $\sum_{L=0}^{M} P(L)$ equals to 1. The leaving process, basically, is a binomial process, with:

$$P(L)=\binom{M}{L}p^{L}(1-p)^{n-L} \quad L=0,1,2,\ldots,M$$

mean: Mp; variance: $Mp(1-p)$

If the number of trials, *M*, is large, the binomial distribution is approximately equal to the normal distribution with mean: Mp and variance: $Mp(1-p)$. As usually in our case the *M* is large (more than 100), we replace this binomial distribution with normal distribution for further calculatingly simplicity.



Above is the graph Binomial distribution vs Normal distribution. Blue represent binomial and red refer to normal distribution. The example is based on mean: $M=100$ *and* $p=0.1$, normal distribution has mean=Mp=10, and variance $Mp(1\text{-}p)$=9, with 2000 Monta Carlo simulated random numbers.

It could be seen from the graph that as *M* approaches large, normal distribution could be functioned as an alternative of Binomial distribution. Consider the fact that most of pension plans have more than 50 active participants, we replace the binomial distribution with normal distribution for further calculation simplicity based on the comparison we made between these two distributions:

$$P(L)=\binom{M}{L}p^{L}(1-p)^{n-L} \quad \Rightarrow \quad P(L) \sim N(Mp,\, Mp(1-p)) \quad L=0,1,2,\ldots,M$$

Coming process : Poisson Process

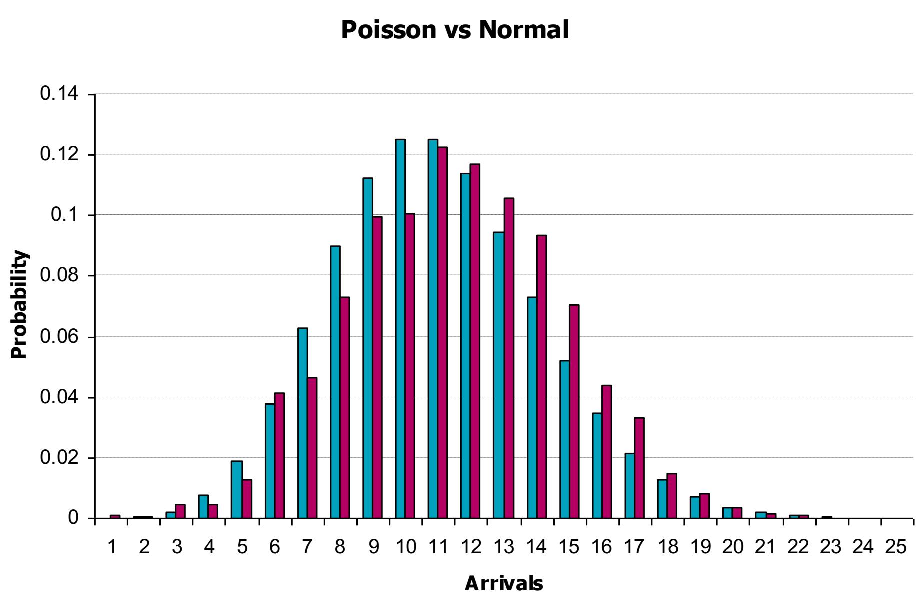
$P(NN = n)$ represents the coming process. The probability tells the likelihood that *n* people coming during time 0 and time 1. The coming process is assumed to be a Poisson process.

$$P(NN = n) = \frac{e^{-\lambda}(\lambda)^n}{n!} \qquad n = 0,1,2,...,k$$

mean: λ

Variance: λ

As the mean number of events per unit of continuum increases, the profile of the distribution resembles that of a Normal distribution. It is relatively simple to calculate P(x) for small values of x, however, for large values of x, the process is complicated by the problems of calculating exponentials and factorials with large arguments. In these cases, it may be desirable to use the normal distribution as an approximation.

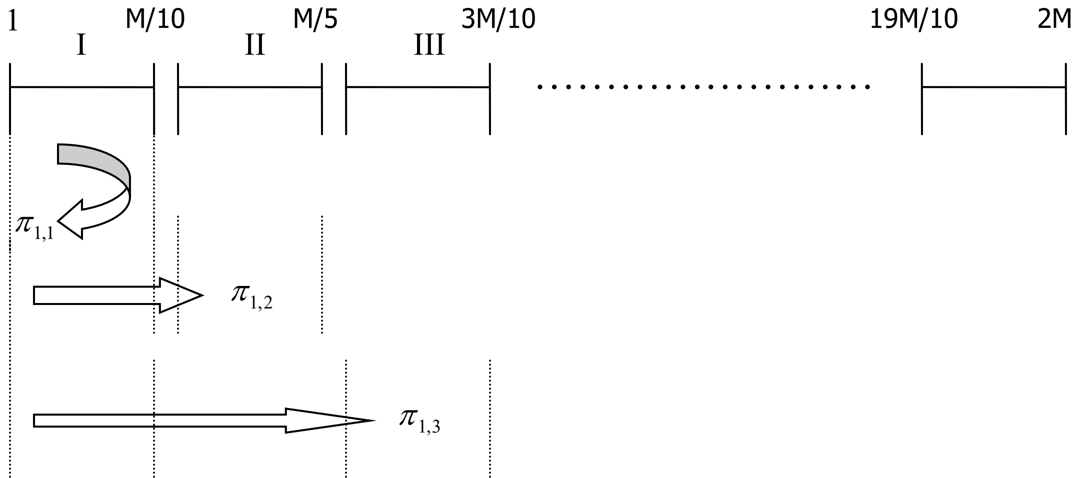


As is showed in the graph, blue represents Poisson distribution vs red represents normal distribution. Normal distribution could roughly be use to approximate poisson distribution in this sense.

$$P(NN = n) = \frac{e^{-\lambda}(\lambda)^n}{n!} \quad \Rightarrow \quad P(NN = n) \sim N(\lambda, \lambda) \qquad n = 0,1,2,...,k$$

Block Function

As of given normalized coming process and leaving process, theoretically it is easy to drive each π_{ij} in the transition matrix. While when back to reality, we simply found this matrix would drive the calculation to far more than complicated since it's a 2M$\times$2M matrix. In this case, $\pi_{M,k}$ would rather been used to describe a block transition instead of individual transition. For instance, based on the former assumption we made, $\pi_{1,1}$ refers to the transition probability that the number of population stay at [1, M/10] from time 0 to time 1, while $\pi_{1,2}$ refers to the transition probability that the number of population jump from [1, M/10] at time 0 to [M/10+1, M/5] at time 1.



Thus $\pi_{I,II} = P(M_I + NN - L = k_{II})$ becomes:

$$\pi_{I,II} = \int_{t=1}^{M/10} P(t \; jump \; from \; I \; to \; II \,|\, t \in I) \times f(t)\, dt$$

$$= \int_{t=1}^{M/10} (P(NN - L) \in [M/5 - t, M/10 - t] \times f(t)\, dt$$

As was demonstrated above, both coming and leaving process could be approximated by normal distribution. This means:

$NN \sim N(\lambda, \lambda)$

$L \sim N(Mp,\ Mp(1-p))$

According to normal distribution,

$(NN - L) \sim N(\lambda - Mp,\ (\lambda + Mp(1-p)))$

By which formula $\pi_{I,II}$ could be calculated. With the same procedure each π can be derived and the block transition matrix can be gotten.

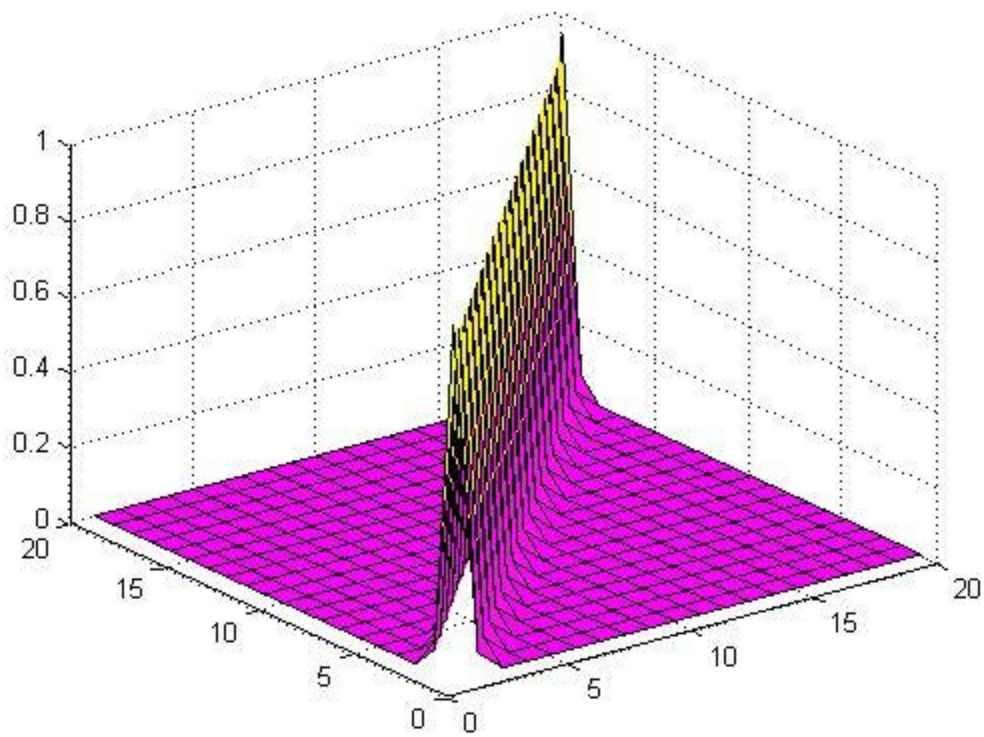
With following assumptions:

M=1000

$NN \sim N(\lambda, \lambda) \sim N(100,100)$

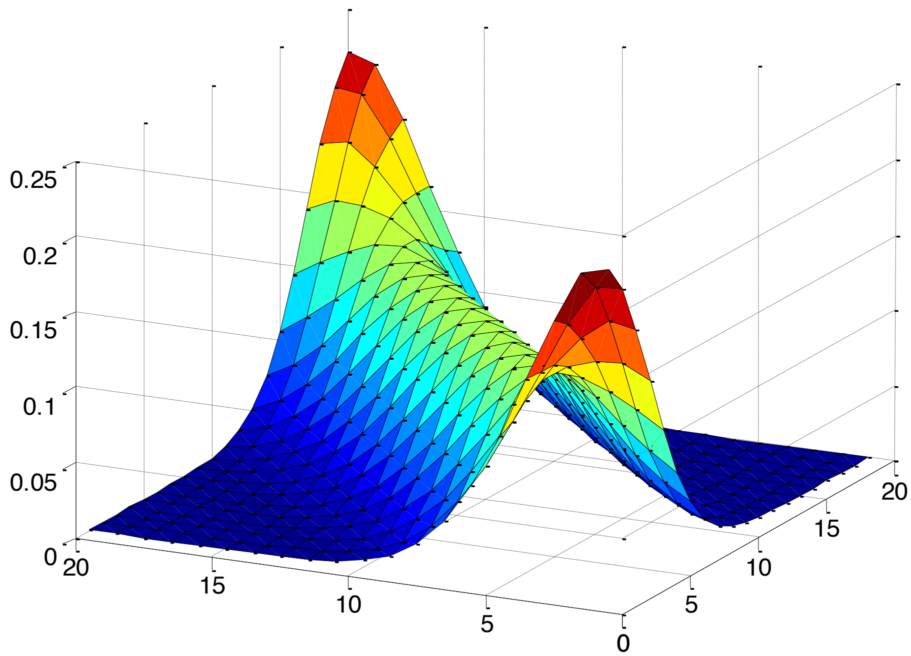
$L \sim N(Mp,\ Mp(1-p)) \sim N(100, 90)$

As showed below, it is a graph of one step transition probabilities:



The third dimension represents the probabilities of each transition. As is implied in the graph, most likely the number of population will stay at the same state $(\pi_{i,j}\ \ i=j)$ instead of moving to another state $(\pi_{i,j}\ \ i \neq j)$.

Π ^ n could be used to calculate the transition matrix for n steps, which, in our case, lead to the below graph demonstrating transition matrix by 20×20 after 100 steps.



In this manner the transition process of population can be estimated.

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