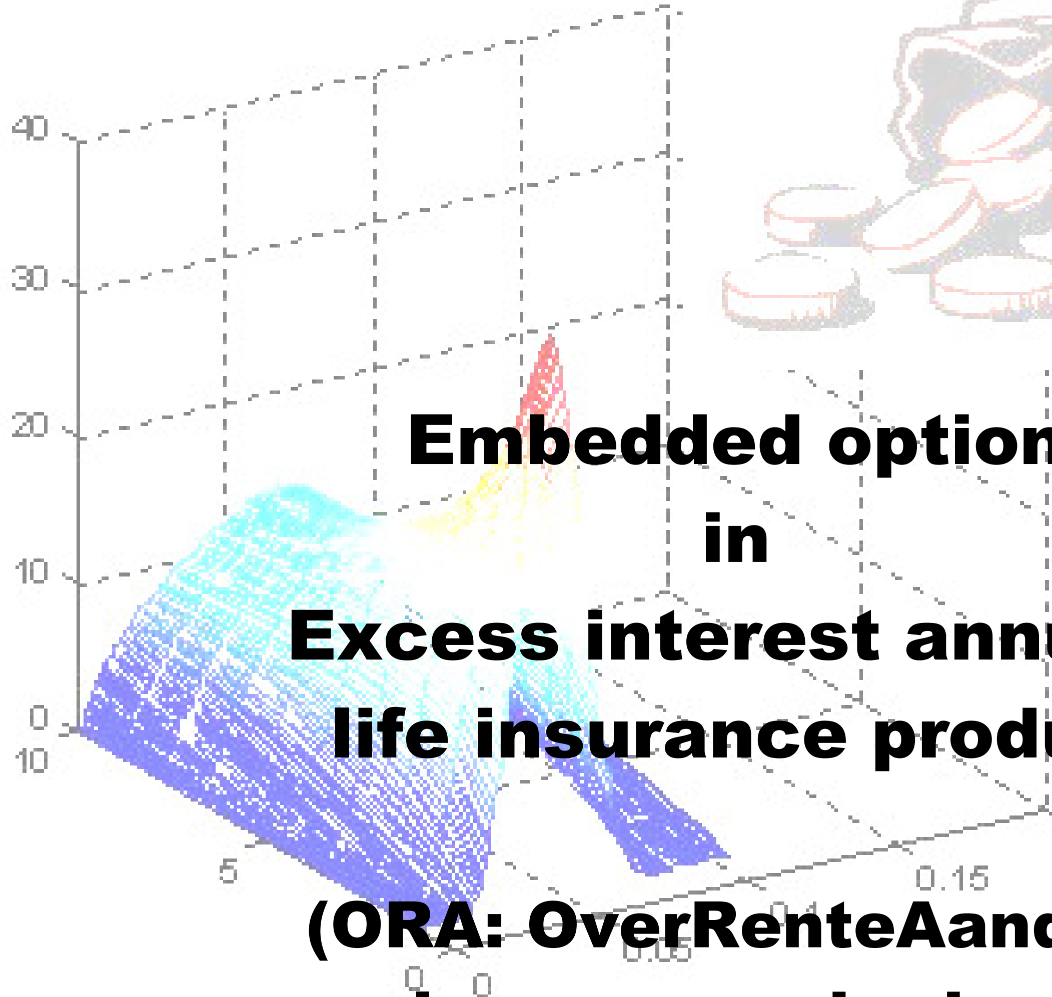
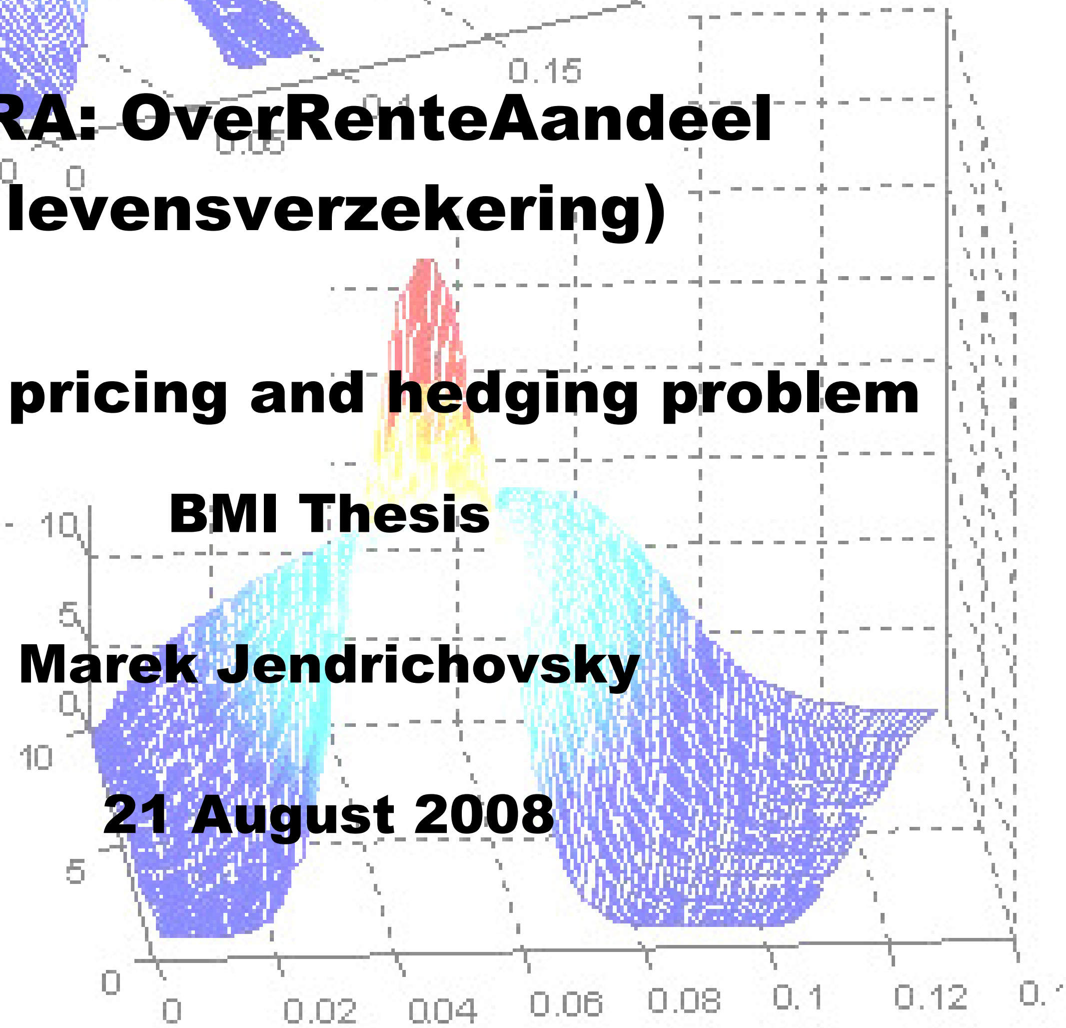


Embedded option in Excess interest annuity life insurance product

(ORA: OverRenteAandeel levensverzekering)

Product pricing and hedging problem

BMI Thesis



Marek Jendrichovsky

21 August 2008

Table of contents:

1 Introduction

The goal of this thesis is basically twofold:

- to analyze the current pricing of the ORA life-insurance product,
- to address the possible “areas of improvements” related to its pricing and hedging.

1.1 Excess interest annuity life insurance product (ORA)

The original Dutch name of this product is: “Overrenteaandeel levensverzekering”, therefore this product will be called ORA-Leven or simply ORA in the rest of the document[1].

The ORA product is form of a *life-insurance* which “insures” the holder of this policy against having no income after his retirement. So *life-insurance* here is actually a little bit too big of a word for a *pension-policy*. To “buy” this policy the insured person pays a *premium* during his active working career, in order to receive, after the “insurance event” (in this case retirement) takes place, regular *annuities* until his death.

What makes this product “special”, is that an insured person receives, above a fixed contractual minimum, a share in the profit based on the difference between a (beforehand fixed) guaranteed interest rate (“rekenrente”) and the yield of the basket of Dutch state-bonds at the time of the benefit (“u-rendement”, further on “u-rate”). It is important to note that the profit sharing here is based on an external benchmark and not discretionary declared by the management of the life insurance company.

This product has characteristics similar to a Constant Maturity Swap (further on CMS) note, see [1]. This is a standard interest rate derivative sold by the larger investment bank. CMS note can be constructed in such a way that each coupon payment depends on a long term rate prevailing in the market at the time of the payment. CMS notes are often constructed in such a way that the coupon payment is floored.

1.2 ORA Product pricing

The purpose of the ORA product pricing is at the moment mainly an estimate for the necessary reserves the bank has to set aside in order to be able to fulfill all its obligations against the insured persons. Therefore the ORA “product”, as seen by the bank’s Assets and Liabilities Management (ALM) department, is actually one huge collection of all the policies the bank sold to its customers.

The price consists of two elements:

- The basic product price, thus the price of the insurance policy without taking the *embedded excess interest option* into account,
- The embedded excess interest sharing option.

The price of the *basic product* is obtained through a straightforward discounted cash-flows calculation, where the only important parameters are the police-life-time expectancy estimations, in order to set-up a time-horizon for the calculation. The pricing of the *embedded option* is not so trivial al all, and as such it constitutes the “sole purpose of the existence” of this thesis.

[1] Another “standard” name for this instrument is GAO: Guaranteed annuity option.

1.3 Sources/References

The general information on the interest rates derivatives pricing could be found in [2], [3], [11] and [12]. The general theory of martingales, stochastic calculus and market completeness is to be found in [4], [5], [6], [7], [10], [9] and [16]. The practical documents about real existing interest rate derivatives are [1], [8], [17] and [18]. The problematic of hedging is subject of [2], [23], [22] and [21]. The Mathematical System Theory is in [24]. More on the Numerical Analysis could be found in [15].

1.4 Business environment

This thesis is written on behalf of SNS REAAL bank, within its BRM department. Balance Sheet and Risk Management (BRM) is a staff department of SNS REAAL. Most important tasks of BRM are:

- measuring and managing risks for the bank and insurance divisions within SNS REAAL,
- capital management,
- liquidity management,
- giving the board of directors of the bank, insurance divisions and the group advice on a framework for optimal value creation.

1.5 Acknowledgements

The author would like to thank his advisors Antoon Pelsser, Andre Ran, Frans Boshuizen and Harry van Zanten not only for helping him to have fun “all the way”, but also helping him to carry the small miseries encountered there. Special thanks go to my SNS-REAAL boss Eelco Scheer who actually offered me the possibility to do my final project at the SNS bank, knowing that my study activities will inevitably conflict in some ways with my performance in his team during my “day job” (as they did). He coped with these tensions in the manner far above my expectations.

2 Interest rate derivatives modeling

The (rigorous/mathematical) pricing of the derivatives (options) is a “100 years old problem”, beginning by the pivotal thesis of *Louis Bachelier* called "Théorie de la spéculation” in 1900.

The first directly applicable result of this field was the so-called *Black-Scholes* formula for the pricing of the European call/put options on stocks in 1973. Black and Scholes could argue that the price of such an option must be calculated as an expected value under a special “risk-neutral/risk-free” probability measure, because in their “arbitrage-free economy” one could purchase a portfolio consisting of a stock and an option on this stock in such a way that this portfolio became risk-free (famous “replicating argument”). The Black-Scholes formula is actually made applicable by Merton, who moved the formula from its original (clumsy) discrete-time setting to the continuous-time settings using the “rocket-science” Ito’s stochastic calculus formula[1].

The further development was directed towards more general setting/derivatives, resulting in so-called “Fundamental Theorem of Financial Mathematics”, stating:

“There is no such thing as a free-lunch” [2]

Stated more rigorously: in the absence of arbitrage there is no self-financing portfolio with zero initial costs V_0 having strictly positive probability of profit $\mathbb{P}(V_T > 0) > 0$ at the end, implying the expected profit $\mathbb{E}V_T$ is also strictly positive, thus:

$$\left(\text{no arbitrage}\right) \Leftrightarrow \left(\neg\exists V : V_0 = 0, \mathbb{P}(V_T \geq 0) = 1, \mathbb{P}(V_T > 0) > 0 \Rightarrow \mathbb{E}V_T > 0\right)$$

Through this condition we see that any $\mathbb{P}-$equivalent measure would “inherit” this arbitrage-free characteristics, as each $\mathbb{P}-$equivalent measure has the same null-sets. Another form of this fundamental theorem states that in the absence of arbitrage, the price of the option is invariant to the equivalent measure change. This means that in the arbitrage-free economy (for sure a reasonable assumption), we can evaluate the price of an option as the expectation under the equivalent measure. The only remaining problem is how to discount the expected value “back” to the present, thus how to calculate the present value of the expected cash-flows of this option. As stated in [10] the discounting must be based on the *opportunity cost of capital*, and this is impossible to quantify for the options as their volatility is dependent on the stock price, continuously moving in time. The achievement of Black and Scholes actually boils down to finding an equivalent *martingale* measure, thus eliminating the drift from their model of the stock price (making the risk-free-rate-discounted stock price a *martingale*), therefore enabling them to evaluate the expectation explicitly and discounting it through the (zero-coupon-bond) risk-free interest rate back to its present value.

---

[1] Recent developments show that Itô, discovering his formula in 1944, was “beaten” by Wolfgang Döblin by 4 years.

[2] Author of this thesis heard this term for the first time at his company’s regular lunch-training-session, as usually called the “free-lunch-session”.

2.1 Equivalent martingale measure example

Finding the equivalent martingale measure can be illustrated nicely in very simple discrete settings, without loss of generality indeed.

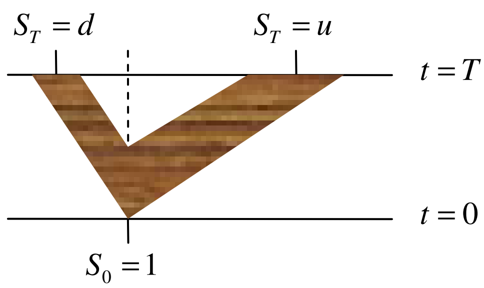
Let's take:

- two-dates economy, thus we can buy today, wait until tomorrow and sell, $t \in \{0, T\}$
- we have (deterministic) zero-interest bond $B : B_0 = 1, B_T = 1$,
- and stochastic stock $S : S_0 = 1$, with two possible "states of the world" tomorrow, thus our (probability) measure space is $(\Omega, \Sigma, \mathbb{P})$:
  $\Omega = \{\omega_1, \omega_2\}, \Sigma = \{\{\varnothing\}, \{\omega_1\}, \{\omega_2\}, \Omega\}, \mathbb{P} : \mathbb{P}(\omega_1) = \mathbb{P}(S_T = u) = p, \mathbb{P}(\omega_2) = \mathbb{P}(S_T = d) = 1 - p$
  meaning price of the stock can go up to u with probability p, resp. down to d with $1 - p$

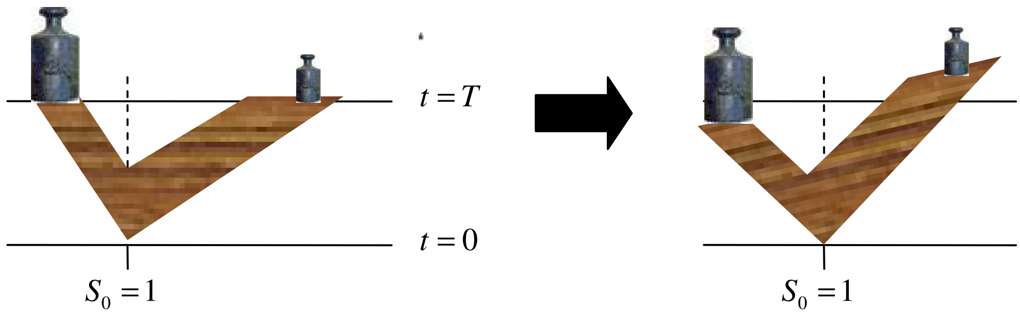
The condition of no arbitrage, equivalently the existence of the unique martingale measure, is that $d < 1 < u$, which seems quite plausible, as a stock going only up would make all investors to abandon bonds and buy only stocks, definitely not a workable model for the market.

We can illustrate this in the "physical world" settings via the following diagrams:

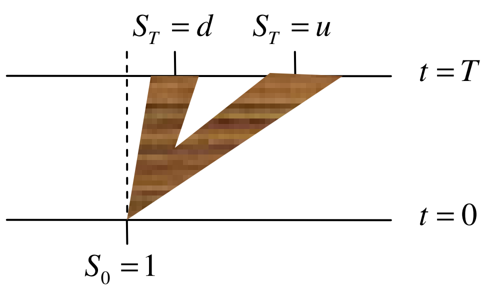
- the problem of finding the equivalent martingale measure in the following physical settings:



- is equivalent with trying to balance-off this V-shaped stock-price-dynamics model with the set of weights (measures):



- this will apparently not work for the situation like this:



2.2 Black's model

A classical approach to price an interest rate derivative is to use *Black's* model. This is actually a "variation on a known theme": the *Black-Scholes* stock-option pricing model, adapted to the options where the underlying is an interest rate rather than a stock.

In contrast with Black-Scholes model Black's model does not describe the evolution of the underlying interest rate y, it merely assumes that this rate is log-normal at the time of maturity T, as well as that its future price equals the spot price at maturity.

Therefore for the logarithm of the rate at maturity, with F the T-forward rate y, F_0 the forward rate y at the time zero, σ the volatility of F and $N(\bullet,\bullet)$ the normal distribution:

$$\ln y_T \sim N\left(\ln F_0 - \tfrac{\sigma^2 T}{2}, \sigma\sqrt{T}\right) \Rightarrow \mathbb{E}y_T = F_0$$

The price of the *call option* on y_T with strike K, and with $P(0,T)$ the current price of the zero-coupon bond maturing at T, would then be:

$$c = P(0,T)\left[F_0 N(d_1) - KN(d_2)\right]$$
$$d_1 = \frac{\ln\left[\mathbb{E}y_T / K\right] + \sigma^2 T / 2}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

The following two aspects can raise some questions about the validity/exactness of the formula:

- contrary to the assumption of the model the forward price instead of the future one is used,
- discounting interest rates are stochastic.

But interestingly enough the effects of these two aspects are cancelling each other out, so the pricing formula contains no approximations, see [2] for more detail.

Nevertheless in order to calculate the prices of more exotic options, like path-dependent ones, Black's model is useless, because is does not model the evolution of the stochastic process for the underlying interest rate.
This issue could be addressed in two ways:

- model the short (spot) rate process and derive the rest of the term structure,
- use real traded interest rates from the market.

2.3 Short rate models

As a typical example of short-rate model we can take the *Hull-White* one factor model. One factor here means one source of the uncertainty, more technically: there is one driving *Brownian Motion* process for all the interest rates.

The model describes the evolution of the short rate r via the stochastic differential equation (further on "SDE"):

$$dr = [\theta(t) - ar]dt + \sigma dW$$

Here a stands for the *mean reversion rate*, σ for the *volatility* and W for *Brownian Motion* (t is time). The function $\theta(t)$ can be calculated from the initial term structure after the parameters a and σ are estimated from the prices of the "calibrating" market-traded (liquid) securities:

$$\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

Here $F(0,\cdot)$ is the initial "forward rate term structure", thus $F(0,t)$ is the current value of the (fictive) interest rate for a loan in the future time t for an infinitesimally short period of time.

Stated formally with:

- T the maturity of the loan,
- $F_M(0,t,T^*)$ the "real" marketed forward rate, available on the market only for some discrete values of T^*,
- $F_M(0,t,T)$ the "smoothed" or inter/extra-polated forward rates with e.g. Nelson-Siegel algorithm,

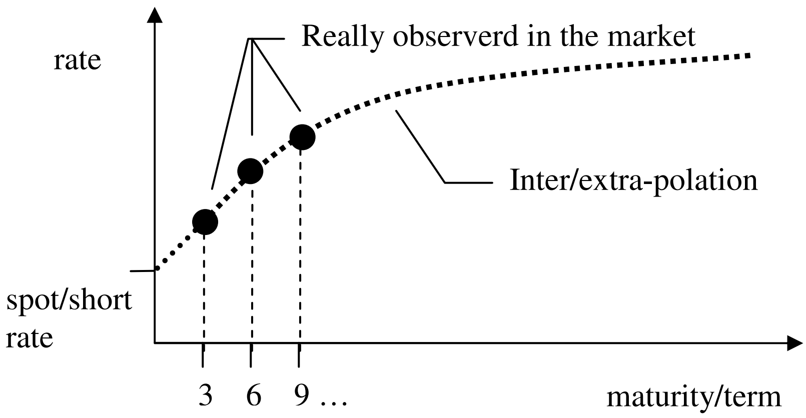
we have $F(0,t) = \lim_{T \to 0} F_M(0,t,T)$.

The relation to the standard term structure (price of the zero-coupon bond $P(0,t)$ with maturity t) is simple and analytical:

$$\begin{cases} P(0,t) = e^{-\int_0^t F(0,s)ds} \\ F(0,t) = -\dfrac{\partial \ln P(0,t)}{\partial t} \end{cases}$$

The advantage of this model is its analytic traceability, the disadvantage its underlying modeling dependence on the "hypothetical"/non-marketed spot interest rates. Hull and White are modeling the evolution of the term-structure (relation describing the dependency of the interest rate, e.g. zero-coupon bond yield, on their maturity/term) through the short interest rate – the hypothetical interest rate with maturity zero.

This could be illustrated by the following example: suppose we can only buy the zero-coupon bonds with the maturities (terms) which are multiples of 3 months, thus the shortest bond we could buy has maturity of 3 months. The term structure, based on the actual market prices, could be drawn[1]:



This disadvantage (model based on the fictive rates) is addressed by the so-called "Market-rate models", which are based on the real market-observed interest rates.

[1] Using some inter/extra-potion scheme like *Nelson-Siegel* one.

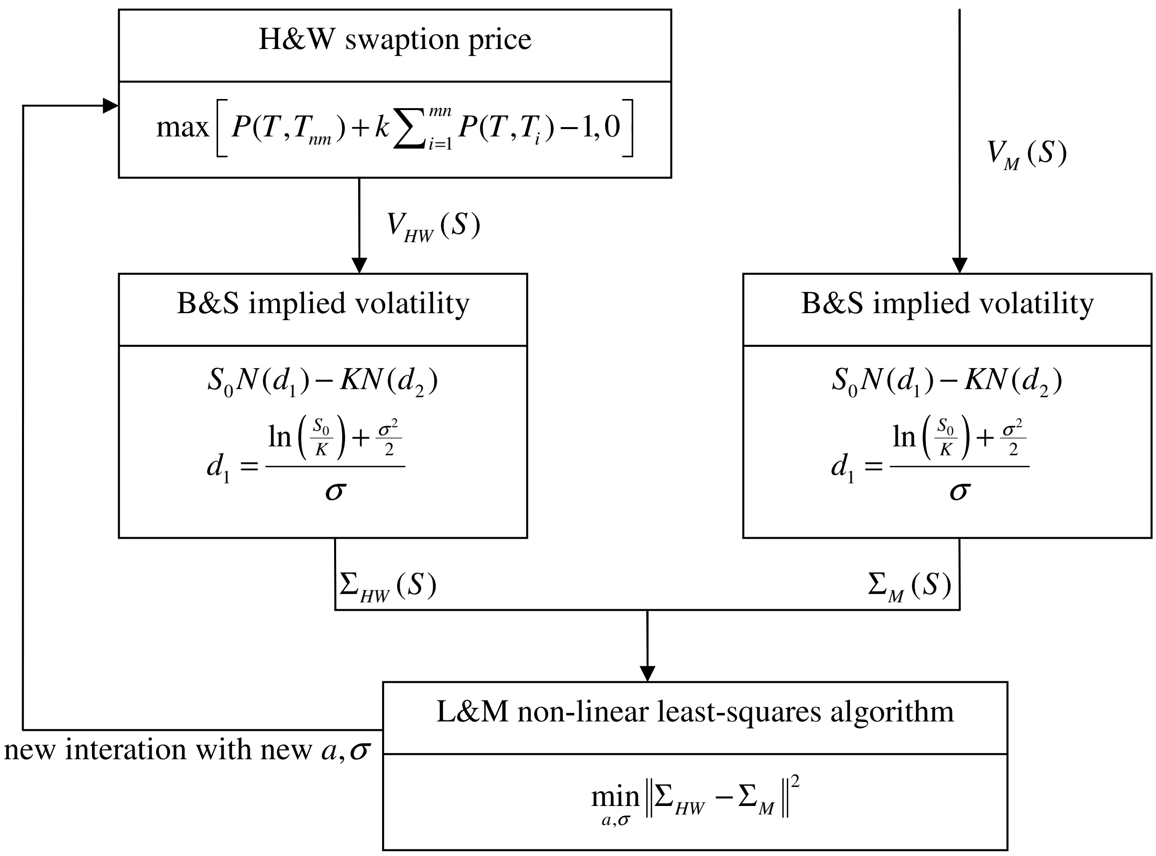
2.3.1 Calibration

The calibration procedure goes as follows:

- we pick some (sufficiently) liquid instruments from the market, e.g. couple of swaptions,
- for these set of instruments we can observe its implied volatility ("feed" its price through the Black-Scholes model),
- for given parameters a and σ we calculate the Hull-White volatility (again "feeding" the Hull-White price through the Black-Scholes model),
- now we "optimize" these two volatilities through the Levenberg–Marquardt algorithm (L&M in the diagram), thus we adjust a and σ and repeat the last two steps.

This we can capture in the diagram where:

- S is a set of N liquid instruments (e.g. swaptions),
- $V(S)$ is the price vector of these instruments (**M**arket or **H**ull-**W**hite), thus $V(S) = \left(V(s_1) \quad V(s_2) \quad \cdots \quad V(s_N)\right)^T$,
- Σ_M is the marketed-instruments volatilities vector: $\Sigma(S) = \left(\Sigma(s_1) \quad \Sigma(s_2) \quad \cdots \quad \Sigma(s_N)\right)^T$,
- Σ_{HW} is the Hull-White implied volatilities vector.



3 Market rate models

The market models try to base the modeling on the existing interest rates, e.g. zero-coupon-bond or LIBOR rates. As a result their pricing formulas are simpler then corresponding formulas from the e.g. short-rate-models as Hull-White.

To illustrate this we show how the market rate model gets rid of the intrinsic problem of the interest rate derivatives pricing: the *discount rate* and the *option's underlying* are stochastic and correlated.

We will use the following terminology/tools:

- $t \in [0,T]$, thus our time-horizon is bounded by some terminal trading date T,
- $r(t)$ is the *stochastic* interest rate,
- $V(t,r)$ is the price of the interest-rate derivative, dependent on time t and rate, r
- $M(t)$ is the "numeraire", our "measurement unit", e.g. bond or money account,
- $\mathbb{P} \to \mathbb{Q}^N$ is the "absolutely continuous change of measure", see [5], actually a *Girsanov* theorem, describing the effects of change from the "natural" probability measure $\mathbb{P}$ into the equivalent martingale measure $\mathbb{Q}^N$, where N stands for the numeraire of this equivalent measure,
- under $\mathbb{Q}^N$ the "numeraire measured" derivative price $\frac{V(t,r)}{N(t)}$ process becomes a martingale, therefore: $\frac{V(t,r)}{N(t)} = \mathbb{E}^N\left[\left.\frac{V(T,r)}{N(T)}\right|\mathcal{F}_t\right] \Rightarrow V(t,r) = \mathbb{E}^N\left[\left.V(T,r)\frac{N(t)}{N(T)}\right|\mathcal{F}_t\right]$

From [3] we can "borrow" the example of the interest-rate derivative pricing, the *Heath-Jarrow-Morton Methodology*, where they perform the equivalent martingale measure change using the bank/money account $B(t)$ as a numeraire, therefore in their methodology the price is:

$$V(t,r) = \mathbb{E}^B\left[\left.V(T,r)\frac{B(t)}{B(T)}\right|\mathcal{F}_t\right] = \mathbb{E}^B\left[\left.V(T,r)\frac{e^{\int_0^t r(s)ds}}{e^{\int_0^T r(s)ds}}\right|\mathcal{F}_t\right] = \mathbb{E}^B\left[\left.V(T,r).e^{-\int_t^T r(s)ds}\right|\mathcal{F}_t\right]$$

and we see how the *correlation* problem manifests itself, as both $V(t,r)$ and the discount term $e^{-\int_t^T r(s)ds}$ are dependent on $r(t)$.

The market model approach is to use different numeraire: the discount (zero-coupon) bond rate $D(t,T)$, thus the price of a zero-coupon bond maturing at T, and then:

$$V(t,r) = \mathbb{E}^D\left[\left.V(T,r)\frac{D(t,T)}{D(T,T)}\right|\mathcal{F}_t\right] = D(t,T).\mathbb{E}^D\left[V(T,r)\middle|\mathcal{F}_t\right]$$

because:

- $D(T,T) \equiv 1$ by the definition of the discount bond,
- $D(t,T) \in m\mathcal{F}_t$, in other words the numeraire is $\mathcal{F}_t$-measurable and independent of $V(t,r)$.

So using the market rate model we can resolve the *correlation* problem. Notice also that $D(t,T)$, the price of the discount bond with maturity T could be straightforwardly obtained from the market (therefore *market-rate* models).

3.1 Change of numeraire theorem

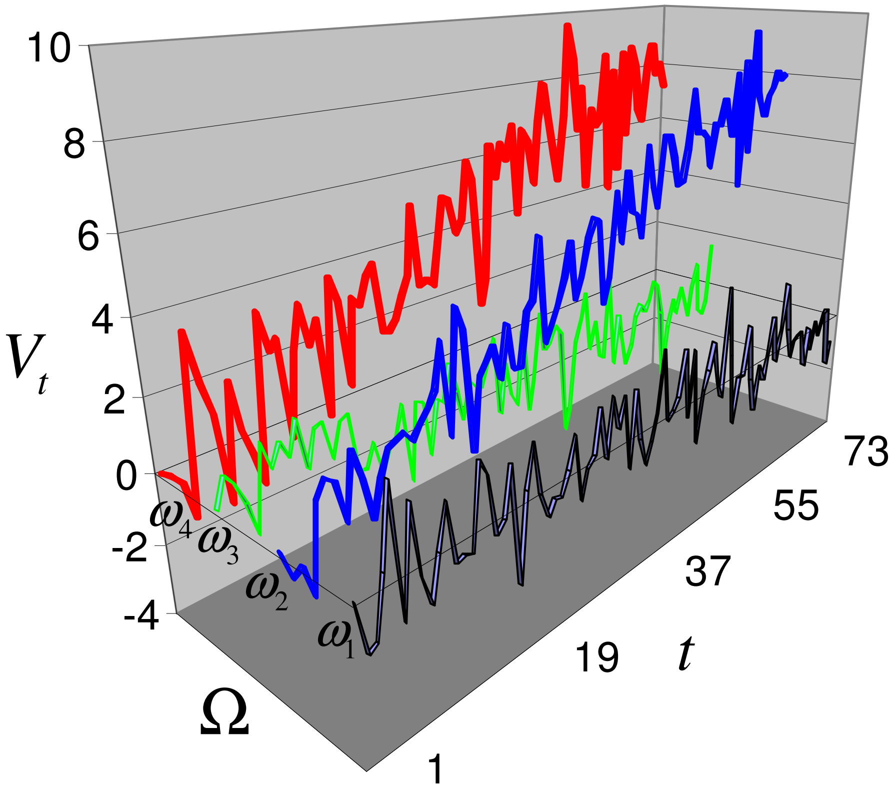
Another important "tool" we will use later is the "Change of numeraire theorem", see [3]. This is actually a "continuation" of the Girsanov theorem (see [5]), adding another (and possibly another and another etc…) measure change.

The Girsanov theorem shows the "effect" of replacing the "natural"/real[1] probability measure of the underlying interest rate $\mathbb{P}$ with the equivalent martingale measure based on the numeraire N , $\mathbb{Q}^N$:

$$\boxed{\mathbb{P}} \xrightarrow{\mathbb{P} \sim \mathbb{Q}^N} \boxed{\mathbb{Q}^N}$$

This martingale property is actually achieved through reassigning the probability mass:

- suppose that for this example (rather restricted, as a typical sample space in question Ω is mostly continuous), that $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \mathbb{P}(\omega_3) = \mathbb{P}(\omega_4)$, implying a positive drift for V_t under $\mathbb{P}$,
- we reassign the probabilities of the individual path under the new equivalent measure $\mathbb{Q}^N$ such that the paths ω_1 and ω_3 get much more probability mass than ω_2 and ω_4 , thus $\mathbb{Q}^N(\omega_1) \gg \mathbb{Q}^N(\omega_2), \mathbb{Q}^N(\omega_3) \gg \mathbb{Q}^N(\omega_4)$, making V_t a martingale under $\mathbb{Q}^N$.



[1] *Real measure* is possibly quite a misleading term here, 'Evidential'/Bayesian (as opposed to 'Frequentist') probability school would call it: "unanimously-agreed-upon subjective measure/belief".

The Change of numeraire theorem “continues” the measure change by adding yet another (and possibly another and another etc…) equivalent measure change, in order to describe the effects of the (further) numeraire change.

Suppose we have the new numeraire $M(t)$, and the underlying interest rate is already expressed through its old numeraire $N(t)$, thus brought under its equivalent martingale measure $\mathbb{Q}^N$, then the change of its numeraire results in:

$$\boxed{\mathbb{P}} \xrightarrow{\mathbb{P}\sim\mathbb{Q}^N} \boxed{\mathbb{Q}^N} \xrightarrow{\mathbb{Q}^N\sim\mathbb{Q}^M} \boxed{\mathbb{Q}^M}$$

Here the price with the new numeraire must of course equal the price of the derivative under the old one, therefore:

$$V(t,r) = N(t)\mathbb{E}^N\left[\left.\frac{V(T,r)}{N(T)}\right|\mathcal{F}_t\right] = M(t)\mathbb{E}^M\left[\left.\frac{V(T,r)}{M(T)}\right|\mathcal{F}_t\right],$$

therefore:

$$\mathbb{E}^N\left[\left.\frac{V(T,r)}{N(T)}\right|\mathcal{F}_t\right] = \mathbb{E}^M\left[\left.V(T,r)\frac{N(T)/N(t)}{M(T)/M(t)}\right|\mathcal{F}_t\right] = \mathbb{E}^M\left[\left.\frac{V(T,r)}{M(T)}.\frac{M(t)}{N(t)}\right|\mathcal{F}_t\right]$$

Thus we see that under the new numeraire $M(t)$ the price can be expressed as the expectation with respect to the corresponding equivalent martingale measure $\mathbb{Q}^M$, with the so-called *Radon-Nikodym derivative* being just the ratio of the numeraires: $\frac{d\mathbb{Q}^N}{d\mathbb{Q}^M} = \frac{M(t)}{N(t)}$, see [3] and [6] for more detail.

3.2 Comparison with spot rate models

We can also make a nice illustrative comparison of the *Market rate models* with the *Spot rate models* like Hull&White:

Model	Relation spot-long rates	Calibration	Valuation formulas	Tractability
Spot	**analytic**; long rate is an integral of the spot rate	**difficult**; based on iterative comparison of implied and market volatilities	complex**; because the spot rate is not a marketed instrument**	**simple** as we have mostly 1-2 factors model
Market	**empirical**; rates simply taken from market	**trivial**	**simple**; market-observed rates directly used	**difficult** because of the multiple factors

3.3 Convexity correction

Sometimes rather exotic non-traded options (e.g. ORA option) could be valuated using the Market models with some "standard"/marketed interest rate, provided that this one gets "corrected" in order to cancel-out the adverse effect of its ("naive" and dangerous) straightforward usage.

This "abusive" usage could include:

- instruments whose pay-off is dependent on the underlying rate directly at the time of its observation (discount bond rate is normally observed at $t = 0$ and paid off at $t = T$, with an exotic instrument the observation and pay-off period could coincide or simply be different from the corresponding bond),
- instruments where the rate is observed in one currency (numeraire), but paid off in another one.

This correction bears the name the "convexity connection" , because, due to the "incorrect" *pay-off time/pay-off currency*, the original linear pay-off (dependency of the price of the option on the underlying price) is being "bent", thus becoming non-linear (convex or concave).

The convexity correction is actually a method where we take liquid/marketed instruments, thus the instruments with more-or-less known probability distributions, and try to estimate the distribution of a more exotic instrument from these.

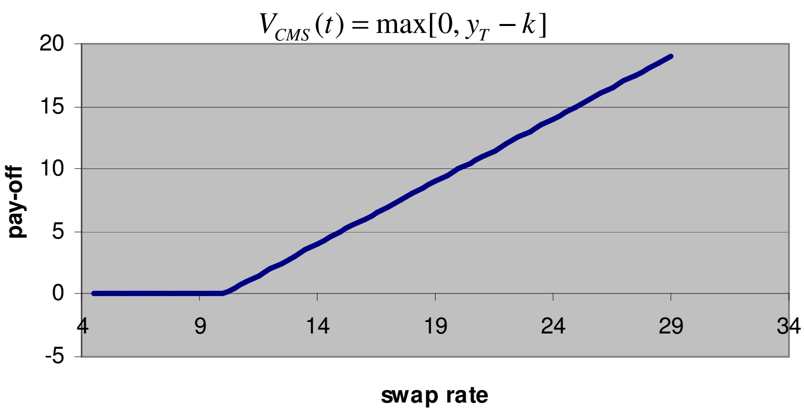
3.3.1 Example using CMS cap

Here we provide the example of the "abuse of the first kind", the "incorrect" pay-off time: the CMS cap product. Standard interest rate swap cap has as a floating leg the LIBOR rate, while the CMS cap derives its value from the e.g. 10-year swap rate. Notice that the standard swap has one floating (LIBOR) leg, and one fixed, while CMS has both legs floating: LIBOR and e.g. 10-year swap rate. The "abuse" in this case lies in the fact that while with the standard swap the (constant) fixed leg is being observed once at the start of the swap and paid off during whole tenor of the swap, while by CMS the 10-year swap rate is observed at the beginning of each period and paid off right-away (more-or-less).

In order to demonstrate the "danger" of applying the 10-year rate without correction, we can use a simple replicating argument: replicate one CMS caplet payoff with one swaption payoff, where the swaption is written only on one period corresponding to this caplet (this period of the CMS cap).

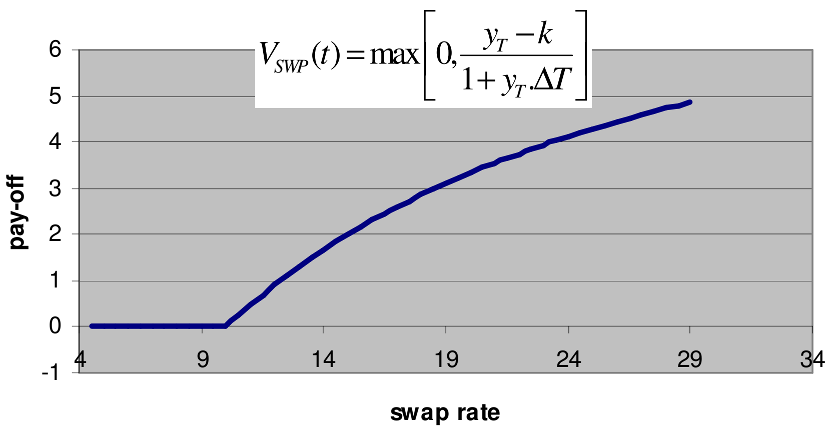
CMS cap pay-off at time T, based on the swap rate y_T and strike rate k, is clearly linear:

CMS caplet pay off



But swaption[1] pay-off at time T, based on the swap rate y_T and strike rate k, is concave, because we value the swaption not at its "usual" time after the period has passed, but right-away at the observation time of the underlying swap rate:

Swaption "incorrectly timed" pay off



Therefore we would have to replicate such a CMS caplet with a *series* of swaptions, see [1].

[1] This "incorrectly timed" swaption's concave payoff is actually equivalent to a more standard "LIBOR-in-arrears" instrument payoff, frequently used to demonstrate the need for a convexity correction.

3.4 Par swap rates

As already mention before one of the main advantages of the Market models is that they make direct use of the existing interest rates.

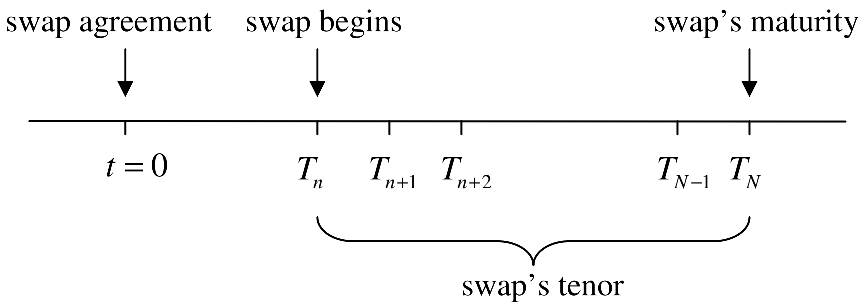
To actually price the ORA option in the market model the u-rate could not be used directly, as it is not a traded security (resp. derivatives with u-rate as underlying). Therefore an analysis (regression analysis) based on the historical data was made and it seems that this rate could be reasonably approximated with the 10-year swap rate.

In the next sections we analyze (among others) “the world of the swap rates”.

3.4.1 Equivalent martingale measure

Let’s look at one such a swap rate and find what we could take as its “natural”/convenient equivalent martingale measure/numeraire.

The standard swap looks like this ($(N-n)$ years swap starting in year n, ending in year N):



It has *floating* and *fixed* legs, where LIBOR is fixed at T_i and paid off at $T_{i+1}, i=n+1...N$. Now the value of the floating resp. fixed leg payment in the $i-$th period is, with D discount bond, α_i the counting convention ($\alpha_i \approx \triangle T$), K the fixed leg rate and $\mathbb{E}^{i+1}$ the expectation under the martingale measure with D_{i+1} as numeraire:

$$\begin{aligned} V_i^{flo}(t) &= D_{i+1}(t)\mathbb{E}^{i+1}[\alpha_i L_i(T_i)] = D_i(t) - D_{i+1}(t) \\ V_i^{fix}(t) &= D_{i+1}(t)\alpha_i K \end{aligned}.$$

therefore the swap pay off is:

$$V_{n,N}^{swap}(t) = \sum_{i=n}^{N-1} V_i^{flo}(t) - \sum_{i=n}^{N-1} V_i^{fix}(t) = [D_n(t) - D_N(t)] - K\sum_{i=n}^{N-1}\alpha_i D_{i+1}(t)$$

Now because we work with the par swap rates, the present value of the swap is zero, and solving $V_{n,N}^{swap}(t) = 0$ for $K = y_{n,N}$ we get:

$$y_{n,N}(t) = \frac{D_n(t) - D_N(t)}{\sum_{i=n+1}^{N} \alpha_{i-1} D_i(t)} = \frac{D_n(t) - D_N(t)}{P_{n+1,N}(t)} \qquad (3.4.1.1)$$

Thus we define $P_{n+1,N}(t) \equiv \sum_{i=n+1}^{N} \alpha_{i-1} D_i(t)$ and call it an *accrual factor/present value of a basis point* or simply PVBP. Therefore each par swap rate $y_{n,N}$ is a martingale under its equivalent martingale measure $\mathbb{Q}^{n+1,N}$ associated with this PVBP numeraire.

3.4.2 Linear Swap Rate Model

As already mentioned in chapter 3.3 some exotic instruments could be priced through more "standard" interest rates, unless they get "corrected". With the ORA option we are facing the "abusive" situation where our ORA pay off is based on the 10-year par swap rate which is being paid off at a "wrong" (earlier/later) time S (ORA option payoff time), instead being paid off "as usual" in periodic terms during 10 years.. In order to evaluate the present value of such "early" or "late" payment, we would have to valuate ORA with respect to the discount bond maturing at S.

Therefore with:

- y_T the 10-year par swap rate at year T, thus in the notation of the equation (3.4.1.1) from the previous section we set $n = T, N = n + 10, t = T$ and thus $y_T \equiv y_{T,T+10}(T)$,
- $\mathbb{E}^S$ the expectation under the martingale measure with S-maturity discount bond as numeraire,
- $\mathbb{E}^P$ the PVBP expectation,

and using the *Change of numeraire* theorem from 3.1:

$$\mathbb{E}^S[y_T] = \mathbb{E}^P\left[y_T \frac{d\mathbb{Q}^S}{d\mathbb{Q}^P} \right] = \mathbb{E}^P\left[y_T \frac{D_S(T)/P(T)}{D_S(0)/P(0)} \right] = \mathbb{E}^P\left[y_T R(T) \right], \qquad (3.4.2.1)$$

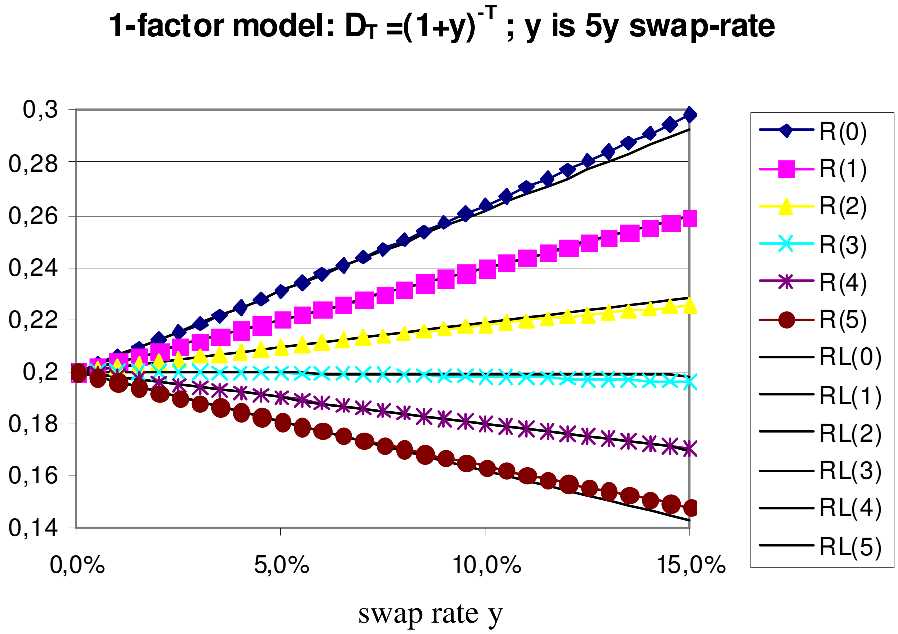
where $R(T)$ stands for the Radon-Nikodym derivative.

This Radon-Nikodym derivative $R(T)$ could be approximated using the *Linear Swap Rate Model*, see [3]. The model approximates $R(T)$ via the linear form:

$$R(T) \approx A + B_S y_T \qquad (3.4.2.2)$$

where $A \equiv \left(\sum_{i=n+1}^{N} \alpha_{i-1} \right)^{-1}$, $B_S = \left(\frac{D_S(0)}{P(0)} - A \right) / y_0$. For our 7-years swap rate with annual payments A would evaluate to: $A = \left(\sum_{i=n+1}^{N} \alpha_{i-1} \right)^{-1} = \left(\sum_{i=n+1}^{N} 1 \right)^{-1} = \frac{1}{N-n} = \frac{1}{7}$.

The Linear Swap Rate Model could be "justified" by the following graph, taken from [11]:



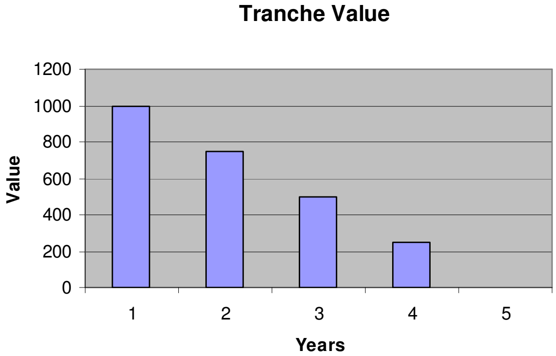
Here we see that indeed the exact Radon-Nikodym derivative $R(T)$, shown for different values of T, is closely approximated by the linear form $R_L(T)$.

4 Valuating ORA with Market rate and Black's models

4.1 The (fictive) tranches mechanism

The embedded option to share the excess profit coming from the higher u-rate is based on the "tranches" mechanism, see [8]. Each year i the incoming premiums are invested in the (new) fictive basket (a tranche) T_i of the 10 year (fictive) u-rate-bonds.

For simplicity (and without loosing any generality) we will show the example of 5 year bond investment: this basket is amortized in 5 equal 25% terms, thus the value of this tranche $T_i(t)$ (fictively) invested in year i, here drawn dependent on year t, looks like this:



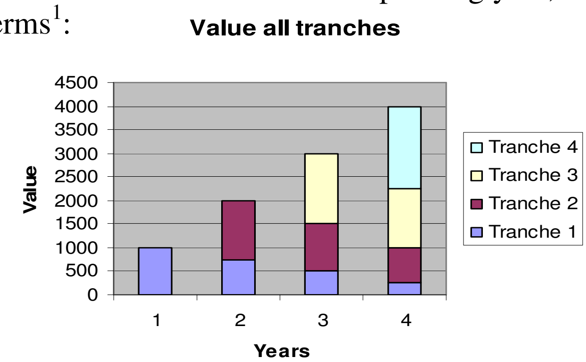
This would be the value of the tranche that has no predecessors, in the "real situation" each tranche (except the first one) has its predecessors, and then the evolution of the tranche values could be described by the following process:

- a (yearly) new tranche is created from the mutations of the reserves (incoming premiums, annuities payouts, redemptions etc…), filled up with the amortizations of its predecessors (thus the amortizations are fictively reinvested),
- the value of the old/predecessors tranches are diminished with the abovementioned amortization,

hence the dynamics are:
$$\begin{cases} T_i(t) = R(t) - R(t-1) + \sum_{j<i} a_j(t) \\ T_j(t) = T_j(t-1) - a_j(t) \end{cases}$$

here $R(t)$ are reserves and $a_j(t)$ the amortization in the year t for tranche i.

Let's now extend the above example to achieve more realistic situation where we have 4 tranches each initiated by the incoming premiums of €1000 in the corresponding year, amortized in the same way in 5 equal 25% terms[1]:



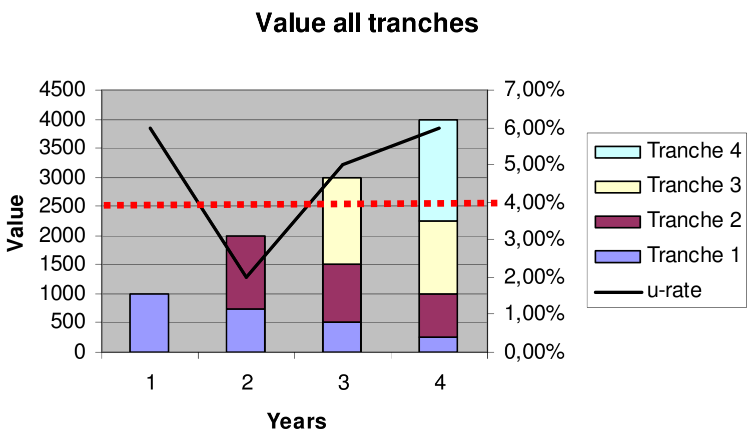
[1] The value in the diagram is the value at the beginning of each year

4.2 The guaranteed rate

The ORA option pay-off at the end of each year is based on the average of the excess u-rates weighted by the value of their tranches. The idea is actually to let the positive excess u-rate in some years (above the guaranteed level) be offset by the possible negative rate difference in other years, and only in the case that the overall balance is positive it is being paid off (this mechanism is called simply the "compensation" further on).

We see that the possible pay off at year t is influenced by the history of the u-rate up to this time, thus the ORA option is actually an Asian (path-dependant) option on the underlying u-rate, see [2] for more details for Asian type options.

Let's enrich our example again with the (fictive) u-rate evolution (the red dotted line is the guaranteed rate of 4%):



And let's calculate the pay-offs based on the guaranteed rate of $k = 4\%$ (dashed red line in the diagram). Notice that for years where the excess u-rate is negative or when the previous years have to be compensated first, we speak of "compensation" towards the customer (negative amount in the table bellow) instead of "payoff" (plus sign), because in case of the "compensation" the seller of the ORA contract "fills up" the "deficient" u-rate for the client so he can receive his guaranteed rate k :

Tranche \ t	1	2	3	4
T_1	1000	750	500	250
T_2		1250	1000	750
T_3			1500	1250
T_4				1750
Payoff/Compensation	**+10**	**-10**	**-5**[1]	**+32.5**

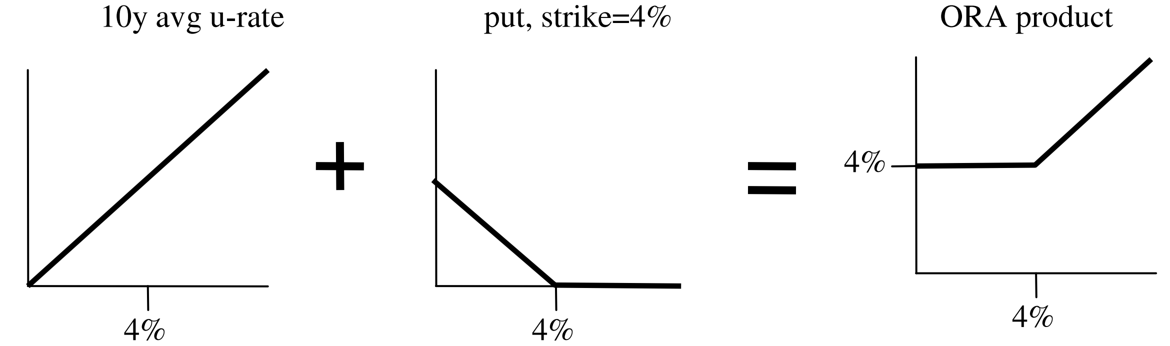
[1] In this year the u-rate was actually above 4%, and a (fictive) profit of 5 was made, but this was not enough to (over-) compensate the accumulated loss of -10 from the previous years. Thus no payoff at the end.

As we will see further on this compensation mechanism is what makes this (already) exotic option a bit too much exotic. If we forget about the mechanism, and consider this ORA product in “standard terms”, we can conclude from the ORA buyer point of view:

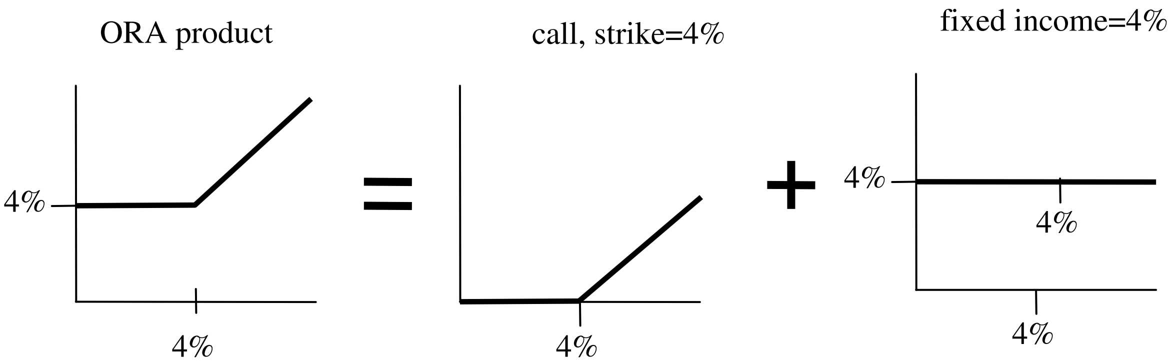
ORA product = 10-year average u-rate note with floor (put with strike=guaranteed rate)

where of course the “right-hand-side” instrument does not really exist on the market.

This we can illustrate nicely through the Bachelier's payoff diagrams “calculus”:



Consequently the pricing of this product could be split into that two, earlier mentioned, sub-products:



In this thesis we will actually consider only this ORA call option.

4.3 The payoff formula

So finally we can state the formula for the option pay-off $V(t)$ at year t, with u_i as u-rate in the year i, $T_i(t)$ the value at the year t of the tranche started in the year i, and k the guaranteed rate, first without taking the compensation in account:

$$V_{nocomp}(t) = \max\left[0, \sum_i T_i(t)u_i - K\right]$$

where $K = \sum_i T_i(t)k$, therefore the pay-off is:

$$V_{nocomp}(t) = \max\left[0, \sum_i (u_i - k)T_i(t)\right] = \left(\sum_i (u_i - k)T_i(t)\right)^+ \qquad (4.3.1)$$

Now we make things "ugly" by incorporating the compensation for the previous (possibly negative) years. We begin by defining the "earnings", the profit/loss made by the fictive portfolio of the tranches in a given year t:

$$E(t) = \sum_i (u_i - k)T_i(t) \Rightarrow V_{nocomp}(t) = \left(E(t)\right)^+$$

When calculating the compensated payoff for a year t we basically face the following couple of situations in relation with the earnings and payoffs in all the years up to t (for simplicity $E(t) \equiv E_t, V(t) \equiv V_t$):

- $E_t \geq 0$, and then if
  - $V_{t-1} > 0$, thus the payoff in year $t-1$ is positive, meaning we have no outstanding losses to compensate, and the payoff function is the simple $V_{nocomp}(t)$,
  - $V_{t-1} = 0$, the payoff in year $t-1$ is zero, meaning we had to compensate already in $t-1$, thus there is still a loss to be compensated, and then the payoff would be: $V_t = \left(E_t - C_{t-1}\right)^+$, where C_{t-1} stands for the outstanding compensation in the year $t-1$,
- $E_t < 0$, the payoff is simply zero, actually again $V_{nocomp}(t)$.

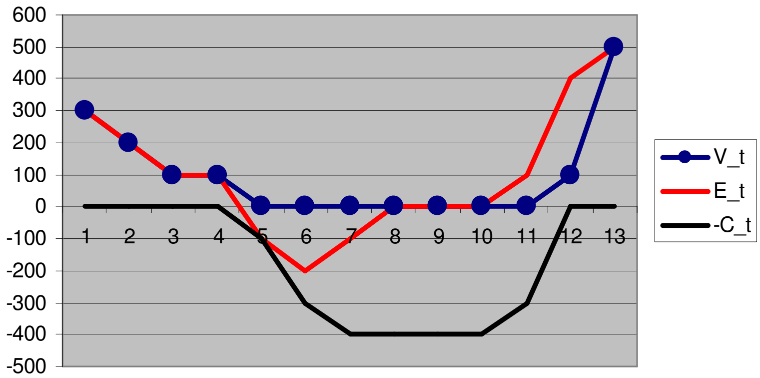
Let's state this more formally[1]: $V_t = \begin{cases} V_{nocomp}(t) & ,\mathrm{V}_{t-1} > 0 \\ \left(E_t - C_{t-1}\right)^+ & ,\mathrm{V}_{t-1} = 0 \end{cases}$

and seeing that $\mathrm{V}_{t-1} > 0$ means that the "outstanding" compensation C_{t-1} must be null, we can simplify this to:

$$V_t = \left(E_t - C_{t-1}\right)^+$$

[1] And "say goodbye" to any hope of arriving at a nice closed formula for the correct compensated ORA option payoff. This kind of recursive formula "begs" for some kind of a computer-run numerical method.

What we have to do now is to derive a (recursive) formula for the outstanding loss-to-compensate (or simply compensation) C_t. The excuse to use recursion is that the possible loss-to-compensate could be compensated by the (possible) positive earnings in the following year, but as soon as this compensation gets "overcompensated", thus the payoff of this following year would be bigger than zero, the (over-) earnings are being paid off, and could not be used to compensate possible future negative earnings. This we can document with a diagram with the payoff V_t, earnings E_t resp. negative of compensation $-C_t$ (for clarity):



Here we see that the compensation C_t is kind of a cumulative sum of the negative earnings, which could be nevertheless "pulled up" by the consecutive positive earnings until it reaches zero again. Thus the "dynamics" of C_t are:

(4.3.2)
$$\begin{cases} C_1 = (E_1)^- \\ C_t = (C_{t-1} + E_t)^- \end{cases}$$

notice also that C_t here is defined here as always positive, as opposed to the diagram, where $-C_t$ is displayed.

This is apparently getting far too complicated for any kind of closed form formula for the ORA payoff, therefore in the current pricing model the equation (4.3.1) is used for actual pricing. Notice then that when using such a formula, the ORA contract holder sometimes receives the excess rate even when it would not be the case with the usage of the "properly compensated" ORA payoff formula. This actually means that the not-compensated ORA payoff formula (4.3.1) assigns too much probability to the positive (non-zero) payoff, and therefore the option gets overpriced. On the other side the seller of such ORA contract has a positive probability of a loss through the compensation paid out to his customers, without any offsetting probability of profit of course, therefore the option price coming from (4.3.1) is "unfair", it would actually create a true arbitrage opportunity in the world where ORA options could be resold and perfectly hedged (we speak about the *complete markets* then). This price is therefore corrected "downwards", as described at the end of the chapter 4.6.

Also as mentioned in 3.4 the u-rate can not be used directly in a market model, therefore a proxy, the 10-year swap rate, is used instead (see [17] for more detail). This rate, as the traded asset (resp. its derivatives), could be modeled from the zero curve, see [8]. Therefore further on we will replace u, the u-rate, with y, the 10-year swap rate

4.4 Levy's method

As already mentioned in 3.4.2, in order to calculate the price of the ORA option in the market model, we have to calculate its payoff at the time S, with the 10-year swap rate y:

$$V(S) = \max\left[0, \sum_{i=S-9}^{S} T_i(S) y_i - K\right],$$

where:

- S is the payoff time,
- $V(S)$ is the option pay-off at year S,
- y_i is the 10-years (par) swap rate observed in the year i,
- $T_i(S)$ the value of the tranche started in the year i at the year S,
- $K = \sum_{i=1}^{S} T_i(S)k$ is the strike, where k is the guaranteed rate,
- and the sum (the average weighted 10-years rate) is calculated only the last 10 years, as the actual amortization scheme used implies that after 10 years a tranche is fully "gone".

In the standard interest rate derivatives models we assume the lognormal distribution for the underlying rate, in this case 10-years swap rate y. In order to value this option we need some estimate of the distribution of the whole weighted sum $\sum_{i=S-9}^{S} T_i(S) y_i$.

This is achieved through so-called *Levy's method* (see [12]): despite the fact that the (weighted) sum of the lognormal random variables **is not** lognormal, we will calculate the first two moments of this sum, and **assume** its "log-normality".

The first moment of the sum could be calculated as follows, assuming independence of the swap rates in the different years, for simplicity we define $T_i(S) \equiv T_i$, because we will not vary S in the following calculations:

(4.4.1) $$M_1 = \mathbb{E}^S[\sum_{i=S-9}^{S} T_i y_i] = \sum_{i=S-9}^{S} T_i \mathbb{E}^S[y_i],$$

where $\mathbb{E}^S$ is already mentioned expectation under the martingale measure from (3.4.2.1).

The question is now how to calculate $\mathbb{E}^S[y_i]$, and the answer could be found in the equivalent martingale world of the par swap rates (PVBP as numeraire),together with the Linear Swap Rate Model.

4.5 Expected swap rate with Linear Swap Rate Model

With Linear Swap Rate Model (3.4.2.2) the expectation in (4.4.1) becomes:

(4.5.1) $$\mathbb{E}^S[y_i] = \mathbb{E}^P\left[y_i R(T)\right] \approx \tfrac{P(0)}{D_S(0)}\mathbb{E}^P\left[y_i(A + B_S y_i)\right] = \tfrac{P(0)}{D_S(0)}\left(\mathbb{E}^P[Ay_i] + \mathbb{E}^P[B_S y_i^2]\right)$$

Now for $\mathbb{E}^P[Ay_i] = Ay_i(0)$, because $y \in \mathcal{M}(\mathbb{Q}^P)$, in plain language y is a martingale under $\mathbb{Q}^P$.

4.5.1 Second moment of the swap rate under its PVBP numeraire

The second term $\mathbb{E}^P[B_S y_i^2] = B_S \mathbb{E}^P[y_i^2]$ is more challenging, but a simple derivation can go along these lines, from:

$$dy = \sigma y dW$$

we know that:

$$y_t = y_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

is just a well-known *Doléan's exponential* indeed (see [5]).

Let's square it:

$$y_t^2 = y_0^2 e^{2\sigma W_t - \sigma^2 t}$$

and take expectation:

$$\mathbb{E}y_t^2 = y_0^2 \mathbb{E}\left[e^{2\sigma W_t - \sigma^2 t}\right]$$

Then use the well-known formula:

$$X \sim N(\mu,\sigma) \Rightarrow \mathbb{E}e^X = e^{\mathbb{E}X + \frac{1}{2}VarX}$$

so we get:

$$y_0^2 \mathbb{E}\left[e^{2\sigma W_t - \sigma^2 t}\right] = y_0^2 e^{-\sigma^2 t} e^{\mathbb{E}2\sigma W_t + \frac{1}{2}Var(2\sigma W_t)} = y_0^2 e^{-\sigma^2 t} e^{2\sigma^2 t} = y_0^2 e^{\sigma^2 t}$$

4.5.2 Second moment the hard way

Completely in the spirit of "why make matters simple if we can make them difficult", we present the original derivation of the second moment[1].

From the *Martingale Representation Theorem* (see [5]) we know that because $y \in \mathcal{M}(\mathbb{Q}^P)$, the stochastic process y could be represented as a stochastic integral with respect to the Brownian motion W, and with its volatility σ:

$$dy = \sigma y dW ,$$

or equivalently expressing this Stochastic Differential Equation (SDE) as the Stochastic Integral Equation:

(4.5.2.1) $$y_t = y_0 + \int_0^t \sigma y_s dW_s$$

This immediately raises a question about how can we estimate the volatility of y, the σ. From [5] we know that the quadratic variance of a stochastic process (a "bracketed" process $\langle y \rangle$) is invariant under a measure change, the volatility under the "original" measure $\mathbb{P}$ is thus the same as under $\mathbb{Q}^P$.

Let's calculate the second moment $\mathbb{E}^P[y_T^2]$ now, thus take expectation of square of (4.5.2.1)**Error! Reference source not found.**, for simplicity we define $\mathbb{E} \equiv \mathbb{E}^P$:

$$\mathbb{E}y_t^2 = y_0^2 + \mathbb{E}[I^2(\sigma y; W)] = y_0^2 + \mathbb{E}\langle I(\sigma y; W)\rangle ,$$

by Ito's isometry (see [5]).

1 "We choose to go to the moon in this decade and do the other things, not because they are easy, but because they are hard"; John F. Kennedy, Rice University in Houston, Texas on 12 September 1962

Solving further:

$$y_0^2 + \mathbb{E}\langle I(\sigma y;W)\rangle = y_0^2 + \mathbb{E}\left[\int_0^t \sigma^2 y_s^2 ds\right] = y_0^2 + \int_0^t \sigma^2 \mathbb{E} y_s^2 ds,$$

by the *Fubini theorem*, see [9], as y_s^2 is a positive measurable random variable.

As we got rid of "stochasticity", we can reformulate this problem as an Ordinary Differential Equation (ODE), with $\psi(t) \equiv \mathbb{E}y_t^2$:

$$\psi(t) = y_0^2 + \int_0^t \sigma^2 \psi(s) ds,$$

and this ODE has only one solution, thus our second moment is:

$$\psi(t) = y_0^2 e^{\sigma^2 T}$$

Let's check this:

$$y_0^2 e^{\sigma^2 t} = y_0^2 + \int_0^t \sigma^2 y_0^2 e^{\sigma^2 s} ds = y_0^2 + \sigma^2 y_0^2 \left.\frac{e^{\sigma^2 s}}{\sigma^2}\right|_0^t = y_0^2 + y_0^2[e^{\sigma^2 t} - 1] = y_0^2 e^{\sigma^2 t},$$

all OK.

4.6 ORA in Black's model with corrected rates

Now back to (4.5.1):

$$\mathbb{E}^S[y_i] = \tfrac{P(0)}{D_S(0)}\left(\mathbb{E}^P[Ay_i] + \mathbb{E}^P[B_S y_i^2]\right) = \tfrac{P(0)}{D_S(0)}\left(Ay_0 + y_0^2 e^{\sigma^2 i}\right) = y_0\left[\frac{A + B_S y_0 e^{\sigma^2 i}}{A + B_S y_0}\right] = \hat{y}_i$$

Now we have the convexity-corrected swap rate $\hat{y}_i$ that we can use for the first moment calculation. Let's see about the second one:

$$M_2 = \mathbb{E}[\left(\sum_{i=S-9}^{S} T_i y_i\right)^2] = \sum_{k=S-9}^{S}\sum_{m=S-9}^{S} T_m \hat{y}_m T_k \hat{y}_k e^{Cov(\hat{y}_m,\hat{y}_k)}$$

If we further assume that all the swap rates have the same (Black's model) volatility σ, and that their (auto) correlation is the same as the one of the *Brownian Motion*:

(4.6.1) $$Cov(\hat{y}_m, \hat{y}_k) = \sigma^2 \min(m,k),$$

then we arrive at:

$$M_2 = \sum_{m=S-9}^{S}\left(T_m \hat{y}_m\right)^2 e^{\sigma^2 m} + 2\sum_{m=S-9}^{S}\sum_{k=S-9}^{m-1} T_m \hat{y}_m T_k \hat{y}_k e^{\sigma^2 k}$$

As already mentioned in chapter 4.3, the compensation mechanism is not incorporated in the ORA option payoff (also called "loss-carry-forward" mechanism), therefore the option gets overpriced at the end. This is compensated via quite an ad-hoc[1] method of reducing the implied Black's volatility[2] σ with 10%, therefore:

$$\sigma \mapsto 0.9\sigma$$

Here we have used the fact the in the Black's model the relation between volatility and option price is injective and "positive", thus by decreasing the volatility used as input for the price calculation, we decrease the price.

Another issue is that the standard swap rate has a certain credit-risk-spread compared to the u-rate, which has no risk-of-default. This justifies another adjustment to the 10-years swap rate to get rid of this credit-risk-spread:

$$y \mapsto y - 0.2\%$$

[1] This reduction was obtained through comparison of the Levy's price with the Monte-Carlo price when the compensation mechanism was taken in account.
[2] This volatility is actually calculated from the implied volatility of the 10-year swap rate observed in the market.

Now with the first and the second moments we are able to calculate the variance of $\sum_{i=1}^{S} T_i y_i \equiv G$:

$$\sigma_G^2 S = \ln\left(\frac{M_2}{M_1^2}\right)$$

And now the final step, becasue with the corrected swap rates $\hat{y}_i$ we can use the standard Black's formula to calculate the present value of the ORA option for one year S:

$$V_S^{ORA}(0) = D_S(0)[M_1 N(d_1) - K N(d_2)]$$

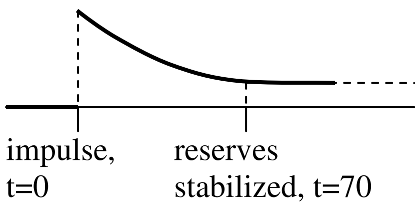
$$d_{1,2} = \frac{\ln\left(\frac{M_1}{K}\right) \pm \frac{1}{2}\sigma_G^2 S}{\sqrt{\sigma_G^2 S}}$$

In order to calculate the present price of the whole ORA option, we have to sum-up the value for each year that option covers:

$$V^{ORA}(0) = \sum_{S=1}^{L} V_S^{ORA}(0),$$

where L could be interpreted as the expected average lifespan of the ORA policy.

The actual used value of L (expected average policy-life-time) is 70 years, which would correspond with the expectation, that an average policy holder enters the contract in his 20s, and terminates it[1] in his 90s, indeed quite an unrealistic assumption. This value of 70 is the result of a different kind of an analysis from the field of the Mathematical System and Control Theory: the *impulse-response* analysis, see [24]. By the system here we mean the status of the reserves of this ORA product: the accumulated sum of incoming premiums, paid off annuities, amortizations and other mutations. By the impulse we mean the (sudden) mutation of these reserves from zero to the "normal"/operational level of 10 tranches with their dynamics.



In other words: after time horizon of 70 years the in/out-coming cashflows are "in balance"/stabilized.

[1] A real euphemism here!

5 Beyond ORA market rate model

In this section we will look at some less clear/"dangerous" issues connected to the current approach of the valuation of the ORA option via the standard Market rate model, in order to derive a more robust approach to the ORA option pricing.

5.1 A serious look at the convexity correction

This section is based on [3], chapter 11.5, but it's being worked-out quite a little bit more here.

We would like to have a look on the convexity correction term in the calculation of the corrected swap rate $\hat{y}_i$:

$$\hat{y}_i = \mathbb{E}^S[y_i] = \tfrac{P(0)}{D_S(0)}\left(\mathbb{E}^P[Ay_i] + \mathbb{E}^P[B_S y_i^2]\right)$$

We see that the convexity correction of the original/marketed swap rate y_i actually comes from the second expectation - second moment of this swap rate y_i. Let's have a (serious) look at this second moment.

With the random variable $Y \equiv y_i$, we denote the value of the original swap rate at the time $i = T$ (further also $\mathbb{E} \equiv \mathbb{E}^P, \mathbb{P} \equiv \mathbb{P}^P$). By assumption of the Black's model Y is lognormal, thus:

$$Y = e^X : X \sim N(\mu, \sigma\sqrt{T})$$

Now in order to state out point more clearly we would like to express Y as a function of the *standard normal* random variable, because we have much more "feeling" about the *confidence intervals* of such a variable:

$$Y = f(\tilde{X}) : \tilde{X} \sim N(0,1)$$

This is what we already know about Y:

$$\begin{cases} \mathbb{E}Y = \overline{y} \\ \mathbb{E}Y^2 = \overline{y}^2 e^{\sigma^2 T} \end{cases}$$

So let's move to this standard normal random variable settings:

$$Y = e^X = e^{X-\mu}e^{\mu} = e^{\sigma\sqrt{T}\left(\frac{X-\mu}{\sigma\sqrt{T}}\right)}e^{\mu} = e^{\mu}e^{\sigma\sqrt{T}\tilde{X}} : \tilde{X} \sim N(0,1)$$

Let's calculate the second moment, actually the *Lebesque integral* over the sample space with respect to the probability measure of Y, and specially the contribution of an subset/"interval" $\{\omega : Y(\omega) \in [V,\infty)\}$ to its value:

$$\int_{\{\omega:Y(\omega)\in[V,\infty)\}} Y^2(\omega)\mathbb{P}(d\omega) = \mathbb{E}[Y^2;[V,\infty)] = \mathbb{E}[\left(e^{\mu}e^{\sigma\sqrt{T}\tilde{X}}\right)^2;[U,\infty)] = \int_U^{\infty} \bar{y}^2 e^{-\sigma^2 T} e^{2\sigma\sqrt{T}\tilde{x}} \frac{e^{-\frac{1}{2}\tilde{x}^2}}{\sqrt{2\pi}} d\tilde{x}$$

where $U = \dfrac{\ln V - \mu}{\sigma\sqrt{T}}$ and using that $e^{\mu} = e^{\ln \bar{y} - \frac{\sigma^2 T}{2}} = \bar{y}e^{-\frac{\sigma^2 T}{2}}$

Actually with $f_Y(y)$ the probability density of Y (as Y is lognormal we know that it exists), plus $g(\tilde{x}) = \dfrac{\ln y - \mu}{\sigma\sqrt{T}}$, and expressing $\tilde{X}$ is terms of Y : $\tilde{X} = \dfrac{\ln Y - \mu}{\sigma\sqrt{T}}$, we have performed an integration by transformation of variables:

$$\int_V^{\infty} y^2 f_Y(y)dy = \int_U^{\infty} g^2(\tilde{x}) f_{\tilde{X}}(\tilde{x})d\tilde{x}$$

But this is then nothing else that:

$$\int_U^{\infty} g^2(\tilde{x}) f_{\tilde{X}}(\tilde{x})d\tilde{x} = \mathbb{E}[g^2(\tilde{X});[U,\infty)] = \bar{y}^2 e^{\sigma^2 T}\phi(-U + 2\sigma\sqrt{T})$$

where in the last step the symmetry of the standard normal distribution around the ordinate axis was used, ϕ is the cumulative distribution function of the standard normal variable.

But now a daunting conclusion might be drawn: for big values of $\sigma\sqrt{T}$, read: large volatility and long maturities, the contribution of the tail of the distribution of Y to the integral:

$$\int_{\{\omega:Y(\omega)\in[V,\infty)\}} Y^2(\omega)\mathbb{P}(d\omega)$$

becomes (too) significant. Let's for example take $U = 2$, meaning that we "operate" on the (quite insignificant, outside of the 95% confidence interval) part of the sample space of $\tilde{X}$ resp. Y :

$$\mathbb{P}(\tilde{X} > U) = \mathbb{P}(Y > V) = 2{,}5\%$$

Now if we take large volatility/long maturity swap rate:

$$\sigma = 10\%, T = 50 \Rightarrow 2\sigma\sqrt{T} = 1.41, \phi(-2 + 1.41) = 0.28$$

we see that the contribution of the "very improbable" outcomes for Y contribute with more that 25% to the level of its convexity correction. In other words, the thickness of the tail of Y, **exactly** where very less outcomes "happen", and from which we thus can't make very robust statistics, contributes a lot to the correction term.

5.2 Levy's method limitations

Remember the closed formula for the ORA option price is achieved through an approximation of the distribution of the sum of the lognormal distributions. This is so called the *Levy's method*, described in [12].

This approximation, which we could express in the shorthand notation as $\left(\sum \text{lognormal}\right) \sim \text{lognormal}$, has of course its limitations, as stated by Levy himself:

- it guarantees only a first-order approximation,
- moreover this assumption is acceptable only for values of $\sigma\sqrt{T}$ less than 0.20

In the previous section we presented an example of the high-volatility/long-maturity swap rate "dangers', where the contribution of the tail was too high compared to the probability of these outcomes, the exact numbers were:

$$\sigma = 10\%, T = 50 \Rightarrow 2\sigma\sqrt{T} = 1.41 \Rightarrow \sigma\sqrt{T} = 0.705$$

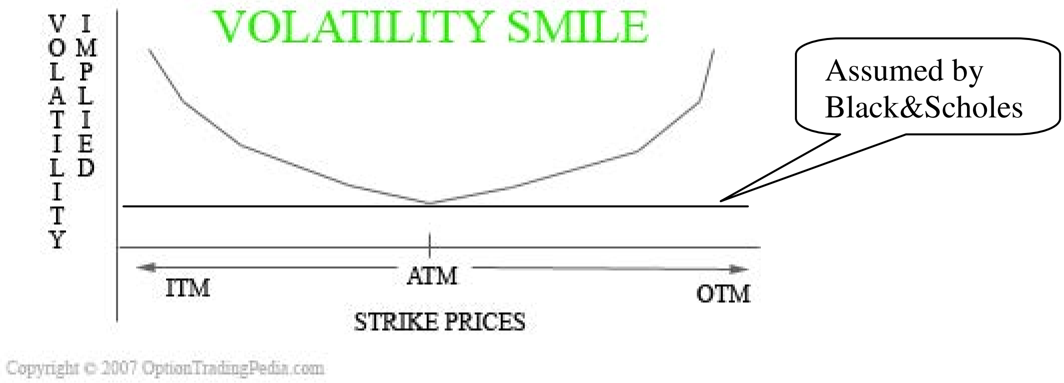
and we see "the finger of Levy rising up"[1].

[1] In December of 2008 the volatilities in question were in the range of 30-35%.

5.3 Volatility smile and implied distribution

Another potential pitfall for all Black-Scholes based valuation models (as the Black's model) is the assumption that the implied volatility of the options is *constant* with respect to the strike level. To this account a very nice quote could be found in [13]:

> **Volatility Smile & The Black-Scholes Model**
> Now, it is obvious that the Volatility Smile chart cannot be plotted without first finding out the implied volatility of the options across each strike price using an options pricing model such as the Black-Scholes Model. However, the resulting Volatility Smile does laugh at the fallacy of the Black-Scholes model in assuming that implied volatility is constant over time. The Volatility Smile shows that, in reality, implied volatility is different across different strike prices even for the same period of time.



Here we see that contrary to the assumption the "In the money" (ITM) and "Over the money" (OTM) options have higher volatility that "At the money" (ATM) options.

In fact we can go even further and ask ourselves a question: "What is then the distribution **implied** by the volatility smile observed in the market?" The answer can be found in [2], where through the simple calculation Hull was able to find such an implied distribution of the European call option with strike K, maturity T, r short rate (constant), g the "risk-neutral"/equivalent-martingale-measure probability density of the stock price at maturity S_T, thus the price is:

$$c = e^{-rT} \int_{S_T = K}^{\infty} (S_T - K) g(S_T) dS_T$$

Let's differentiate with respect to K:

$$\frac{\partial c}{\partial K} = -e^{-rT} \int_{S_T = K}^{\infty} g(S_T) dS_T$$

And again the same:

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K)$$

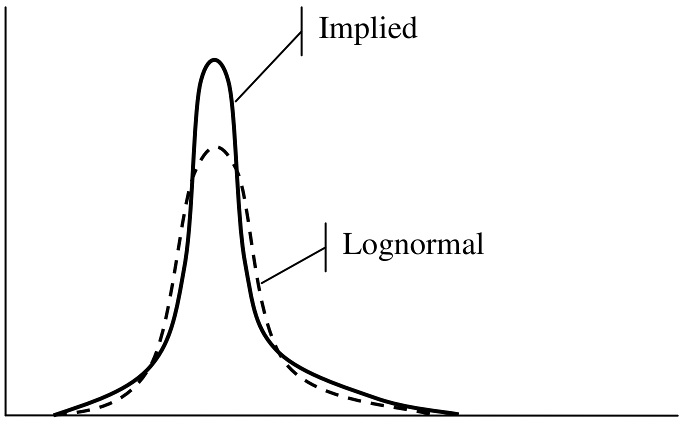
Therefore:

(5.3.1) $$g(K) = e^{rT}\frac{\partial^2 c}{\partial K^2}$$

Of course, normally we have no analytical expression for c based on the market data, as only a limited set of strikes are normally available for certain instrument, but we can approximate, say we have 3 strikes: $K-\delta, K, K+\delta$, priced at c_1, c_2, c_3, then we can estimate the value of g at K with the standard numerical analysis 2nd derivative approximation as:

(5.3.2) $$g(K) = e^{rT}\frac{c_1 + c_3 - 2c_2}{\delta^2}$$

Now the question is what comes out of the market as the implied distribution? And the answer is less then ideal in connection with the previous two sections. It is exactly the **tail** of the distribution which gets **thicker** based on market evidence. Thus the convexity correction in the market rate model for such distribution would be quite off the mark.



5.4 Limitations of the ORA market rate model

Let's sum up the limitations of the current ORA pricing model:

- *approximation* of the u-rate with the 10-year swap rate proxy,
- *estimation* of the thickness of the tails of the swap-rates distributions based on the very "improbable" outcomes,
- *approximation* of the Radon-Nikodym derivative in the convexity corrected rate with the *Linear Swap Rate Model*,
- *approximation* of the sum of the lognormal rates with another lognormal random variable,
- *adjustment* in the implied volatility in order to cancel the overpricing of ORA option by not incorporating the compensation mechanism,
- *assumption of the autocorrelation structure* of the swap rates in order to be able to calculate the expectation with their joint distribution.

Actually all this serves a sole purpose of estimating of the unknown (possible joint) distribution of the "exotic" instrument through the distributions of less complex, more standard instruments. To make the process more manageable the current ORA model created all the above assumptions about the information we simply miss from the sought distribution.

Basically when we are trying to improve the current ORA pricing/hedging model, we are facing two basic issues:

- good fit to the risk neutral (implied) distributions of the individual swap rates,
- and their autocorrelation structure.

From ref [3] we know that the first problem is not so critical for the pricing problem, among the models studied in [3] the marginal distributions fit was a much lesser of these two problems.

The second problem is much more "intrinsic", here inevitably we will have to use some assumptions about the autocorrelation again, but we would like to do it in a much more controlled way. To study this autocorrelation problem in more depth, a serious look at the theory of the market (in-) completeness and hedging is required, as provided in the following chapter.

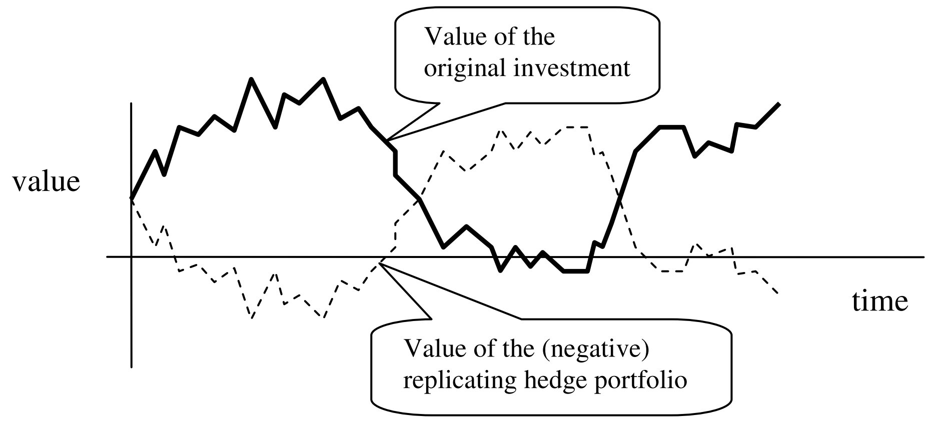
6 Hedging

6.1 Why hedging

A bank selling an option to its customer faces a situation where it makes quite a large chance of making or loosing a lot of money with (relative to the underlying asset) small initial investment. Not all the banks want to take such a risk, and therefore they try to control the amount of risk by letting other parties take over part of it, together with the corresponding extra profit/loss bound to this ("piece of") risk.

Such a bank can employ the following strategies in dealing with such risk:

- do nothing, this is so-called *naked*[1] position,
- buy the option's underlying asset (in case of a call option), and thus assume *covered* position,
- some ad-hoc method like a *stop-loss* strategy, see [2] for more,
- use *hedging*, essentially replicating the investment by buying the portfolio which "mirrors" the movements of the original investment:



But before we look into how exactly this last strategy – hedging works, we must first check if it is really possible to (perfectly) replicate any financial claim in a given market model, or in the real market.

[1] Risking that the market, through its adverse movements, assumes the role of that small child in the famous tale of H.Ch. Andersen, screaming: "The king is naked!"

6.2 Complete/incomplete markets

A nice parable to the incomplete markets is given in ref [16], where a couple from the well known nursery rhyme[1] is considered: the man eats no fat and his wife no lean and to top it all they get divorced, creating a separate demand for lean and fat in the "economy" that knows only one single product: pork chops consisting of one portion of fat and lean respectively. This economy/market is thus *incomplete*, because there is no way how somebody can buy fat (resp. lean) separately, in other words: fat resp. lean can't be "replicated" in that given market from other assets. This requirement for the *completeness* is actually called "spanning the market": this word spanning actually comes from the mathematical field of *Linear Algebra*, where we consider the market as a vector space, where vectors (assets) can (or can not) be replicated by other vectors (assets) through their linear combination (or a *stochastic integral* in the more realistic settings of continuous trading and random claims "vector space").

An example from these settings could be found in ref [20], where the authors consider different incomplete markets, one for example the 2-factor Black and Scholes model, the so called "stochastic volatility" model:

$$\begin{cases} dP_t = \mu P_t dt + \sigma_t P_t dW_{P_t} \\ d\sigma_t = g(\sigma_t)dt + \kappa\sigma_t dW_{\sigma_t} \end{cases}$$

where the (in Black and Scholes model constant) volatility σ is here driven by a different (not completely correlated, 2nd factor) Brownian Motion compared to the asset P. The incompleteness of this model comes from the *Martingale Representation Theorem* (see [5]) which states that only in the market driven by the simple factor (1-dimensional Brownian Motion) every random variable "living" on the terminal time T σ-algebra $\mathcal{F}_T$ (a random payoff of a contingent claim at its maturity) could be expressed as a stochastic integral of the trading strategy with respect to this Brownian Motion.

The question of market (in-) completeness is actually a very subtle one: we can model (essentially) incomplete market as if it was complete. A nice (and close) example is the current ORA option pricing model, remember that the autocorrelation of the swap rates was defined in (4.6.1) as:

$$Cov(\hat{y}_m, \hat{y}_k) = \sigma^2 \min(m,k)$$

This assumes that the autocorrelation of the swap rates follow a fairly simple, straightforward and above all deterministic process, while in reality we know that there's no separate asset (the "fat" from our parable) which would derive its price from the autocorrelation of the swap rates, and therefore there is simply no information in the market about the implied autocorrelation structure aside from (brutally) simple empirical autocorrelation estimated from the historical data. In other words: in the current ORA option model we have replaced the unknown (and very probably complex) autocorrelation with a simple one, and hence we can pretend that the ORA-market-model is complete. The lack of this autocorrelation market information also implies that there is no use to employ some fancy method like "Maximum Entropy Joint Distribution", see [25], because the information/entropy is simply not there.

[1] "Jack Sprat could eat no fat / His wife could eat no lean / And so betwixt them both / They licked the platter clean!"

6.3 How to hedge

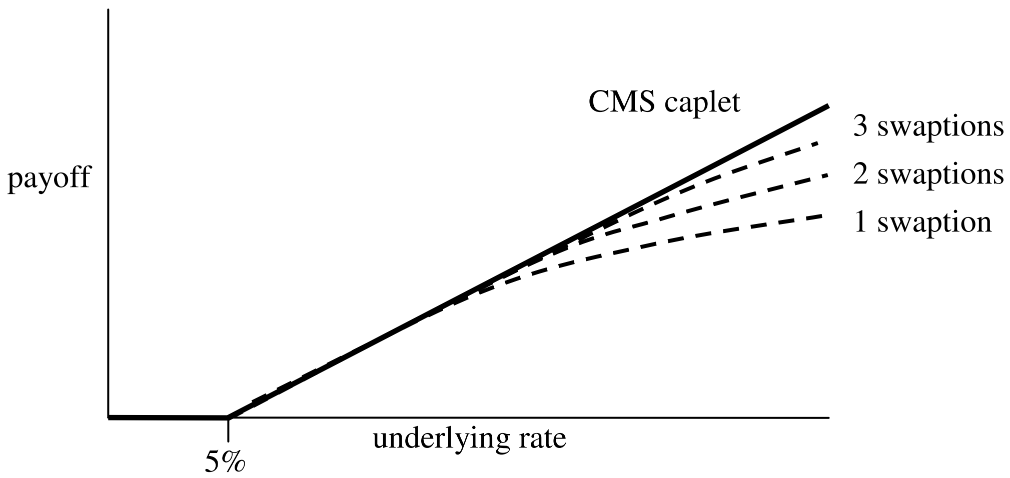
The hedging actually means protecting one's investment against movements in the underlying variables. This could be achieved via taking the opposite position in the assets that depend on the same underlying. The opposite position means that our hedging portfolio should "react" on the changes in these underlying variables exactly in the opposite way as our liabilities, effectively making our hedged investment insensitive to the underlying movements. This hedging portfolio/strategy could take one of the two forms:

- *static*,
- *dynamic*.

6.3.1 Static hedge

Static hedge portfolio is a basket of the plain/vanilla assets that are supposed to replicate (to some degree) some more complex instrument's payoff. This is also called "hedge and forget", because once the hedge is set up, there's no need to re-adjust it until the payoff date. Nevertheless if some exotic option, like a barrier option, is replicated by a static hedge, the intermediary replicating plain vanilla options written on the dates prior to the maturity (on the barrier boundary), must be unwind "along the way". For more details see [2].

Here we present a concrete example from [1], a replicating static hedge of a CMS note caplet, with (consecutively) 1,2 or 3 plain swaptions:



We see here that the accuracy of the static hedge grows only together with its complexity. We also have to remark that the notionals (in their sum equal to the notional of the CMS note caplet) of these plain swaptions are very disproportional, the first swaption gets almost all the "mass", the rest is substantially smaller.

6.3.2 Dynamic hedge

Dynamic hedging could be mathematically best illustrated by means of the *Taylor series expansion* of the investment value, in this example the investment is a short-selling of the call option on the stock: with the current stock price S_0, time t_0, volatility σ_0 and risk-free rate r_0 we can express the value C of the call option written on this stock with the help of the Taylor series for C around $(S_0, t_0, \sigma_0, r_0)$:

$$C(S,t,\sigma,r) =$$
$$= C(S_0,t_0,\sigma_0,r_0) + \frac{\partial C}{\partial S}(S-S_0) + \frac{\partial C}{\partial t}(t-t_0) + \frac{\partial C}{\partial \sigma}(\sigma-\sigma_0) + \frac{\partial C}{\partial r}(r-r_0) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S-S_0)^2 + \mathcal{O}(t^2,\sigma^2,r^2,S^3)$$

Then by simplification $\Delta x \equiv (x - x_0)$ and leaving the higher order term $\mathcal{O}(t^2,\sigma^2,r^2,S^3)$ out, hence *approximating* C around S_0:

$$\Delta C \approx \frac{\partial C}{\partial S}\Delta S + \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial \sigma}\Delta\sigma + \frac{\partial C}{\partial r}\Delta r + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left(\Delta S\right)^2$$

The individual partial derivatives (and corresponding dynamic hedging strategies) of the call option price are called the *Greeks*, and with them the change in the option value could be approximated by (we leave Δ's from the previous approximation out, because the first partial derivative is known under the same Greek letter delta: $\Delta = \frac{\partial C}{\partial S}$):

$$C \approx \Delta S + \Theta t + \nu\sigma + \rho r + \frac{1}{2}\Gamma S^2$$

Now as apparently it doesn't make sense to hedge against the "flow of time"[1] (thus "theta-hedging" Θ), the following (mixtures) of dynamic hedging strategies are currently used:

- Δ, *delta* hedging, protects the investment against small (or linear) movements in the underlying asset,
- ν, *vega* hedging, volatility movements protection,
- ρ, *rho* hedging, softens the effects of the changes in the short, risk-free rate
- Γ, *gamma* hedging, protects the investment which Δ is very sensitive to the changes in the underlying asset, and the pure delta hedging would need to be done very frequently to guarantee the stability.

Now it's clear that to hedge one's position dynamically means *continuously* taking the opposite position in the underlying assets based on the (ever changing) Greeks. Actually this Taylor-series analysis of hedging also demonstrates the impact of the discrete-hedging (as the true continuous hedge is only an "ideal") on the efficiency of the hedge.

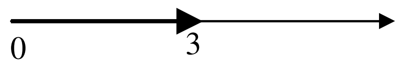
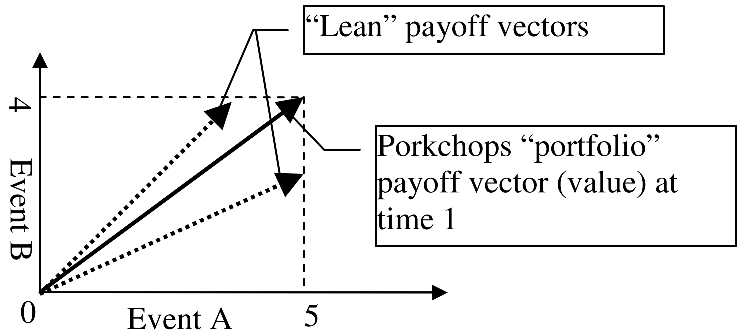
[1] Most attempts in the field of plastic surgery performed on the rich and aging ladies only support this view.

6.4 Hedging and completeness

It's not surprising that the Geometry was the first mathematical discipline to develop in the ancient Egypt and Greece; the Chinese also considered it to be better than 1000 words. In this section we will illustrate the relation between hedging and the market's completeness through geometry, taking two examples.

6.4.1 Static replicating/hedging: the "pork-chops economy"

This is the economy from the chapter 6.2, with its very limited variety of the products. We will illustrate the impossibility of statically replicating the "lean" product in the economy which knows only the pork-chops consisting of lean and fat together. Our economy knows two dates: 0 =now, 1=the maturity/payoff date, and two events: A =price of lean at time 1 goes up, B =price of fat at time 1 goes up. Notice how the payoff space gain dimensions when there are evens being added to the filtration. The performances of the "investment" in this diagrams are represented by the payoff vectors, e.g. the payoff vector of the pork-chop portfolio, which has value 3 at time 0, grows to 5 in case that event A happens and to 4 if B at time 1, thus the payoff vector is $(5,4)$:

Filtration:	**Payoff space:**
$\mathcal{F}_0$	 Porkchops "portfolio" payoff vector (value) at time 0
$\mathcal{F}_1 = \{A, B, \ldots\}$	

Now we see that we can't replicate a portfolio consisting purely of "lean" product, because such product would have payoff (with the initial value of 3) of (5,3) in case of A and (3,4) if B . We see that this has a nice geometric interpretation as:

$$\left\{\begin{pmatrix}5\\3\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix}\right\} \notin \text{Span}\left\{\begin{pmatrix}4\\5\end{pmatrix}\right\}$$

where (4,5), the pork-chop payoff, is the only payoff vector possible in our limited economy.

6.4.2 Dynamic replicating/hedging: the “Black-Scholes economy”

In this section we move to slightly more complex (and realistic) setting:

- we have a 3-date economy:
  - 0=today,
  - 1=tomorrow,
  - 2=payoff time,
- two products:
  - (stochastic) stock: price at time 0..2 is $S=\{S_0,S_1,S_2\}: S_0=3$,
  - (deterministic) bond B : with risk-free rate=0%, $B=1$,
  - (stochastic) call option on S , strike 3, maturity 2: $C=\{C_0,C_1,C_2\}: C_0=2{,}5$, actually the portfolio we would like to replicate,
- therefore we model this through:
  - measurable space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{Q})$, where:
  - sample space $\Omega=\{\omega_1,\omega_2,\omega_3,\omega_4\}$,
  - filtration $\mathbb{F}=\{\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2\}$,
  - (risk-free) probability measure $\mathbb{Q}(\{\omega_i\})=\frac{1}{4},\forall i$,
  - events:
    - A=stock goes up to 7 at time 1,
    - B=it goes down to 3 at 1,
    - C=up to 9 from 7 at 2,
    - D=down to 5 from 7 at 2
    - E=up to 5 from 3 at 2,
    - F=down t 1 from 3 at 2.

From these events we can calculate the price of the call option and the replicating portfolio process, but first we check that our probability measure $\mathbb{Q}$ is indeed risk free, so let’s check if our (B -discounted) stock process S is a martingale under $\mathbb{Q}$ (with $B\equiv 1$ as numeraire):

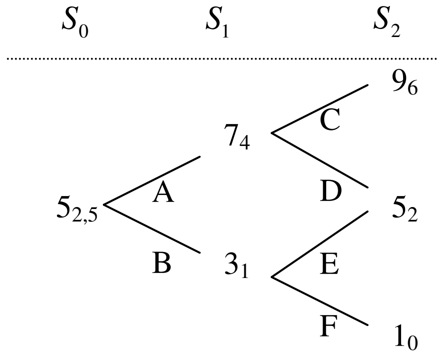
$$\mathbb{E}[S_1 \mid \mathcal{F}_0]=7.\mathbb{Q}(A)+3.\mathbb{Q}(B)+=7.(\tfrac{1}{4}+\tfrac{1}{4})+3.(\tfrac{1}{4}+\tfrac{1}{4})=\tfrac{10}{2}=5=S_0$$

$$\mathbb{E}[S_2 \mid \mathcal{F}_1]=\begin{cases}7=S_1, & \text{if A}\\ 3=S_1, & \text{if B}\end{cases}$$

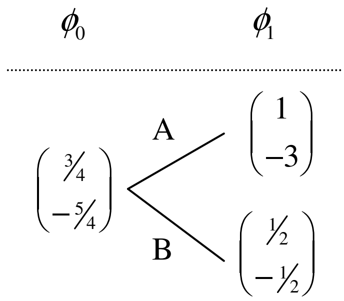
$$\mathbb{E}S_i=5<\infty,\forall i$$

and we see that all the conditions are satisfied, therefore $S\in\mathcal{M}(\mathbb{Q})$.

Now we calculate backwards the payoff (and thus also value) of the call option: $C = \max\{0, S_2 - 3\}$, through a binary tree, option's payoff is the subscribed number bellow the stock price:



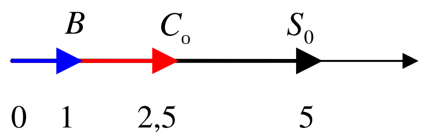
The following diagram describes the call option replicating portfolio $\phi = \{\phi_0, \phi_1\} : \phi_t = \begin{pmatrix} \phi_t^{(0)} \\ \phi_t^{(1)} \end{pmatrix}$. The replicating actually works through adjusting the replicating portfolio at each step t in order to contain $\phi_t^{(1)}$ units of stock and $\phi_t^{(0)}$ units of bonds, thus the value $\phi_0 = \begin{pmatrix} 3/4 \\ -5/4 \end{pmatrix}$ means that at time zero we have to buy $3/4$ of stock and lend $5/4$ of bonds. The replicating portfolio should replicate the payoff of the call option, therefore in continuous settings we could state: $\int_0^t \phi_s dA_s = C_t : A_t = \begin{pmatrix} B \\ S_t \end{pmatrix}$:



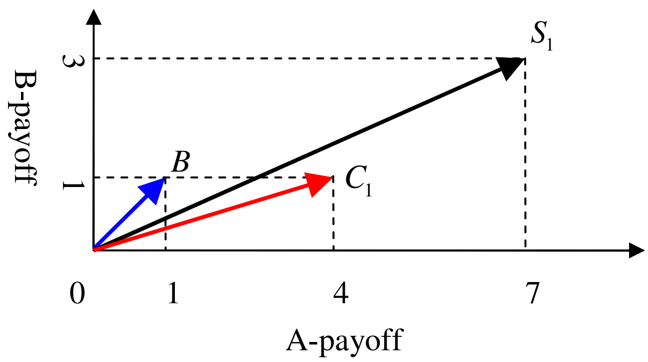
Let's draw the evolution of this model now (scaling is only approximate), the payoff vectors for the call option C and the replication portfolio ϕ are identical:

Filtration: **Payoff space:**

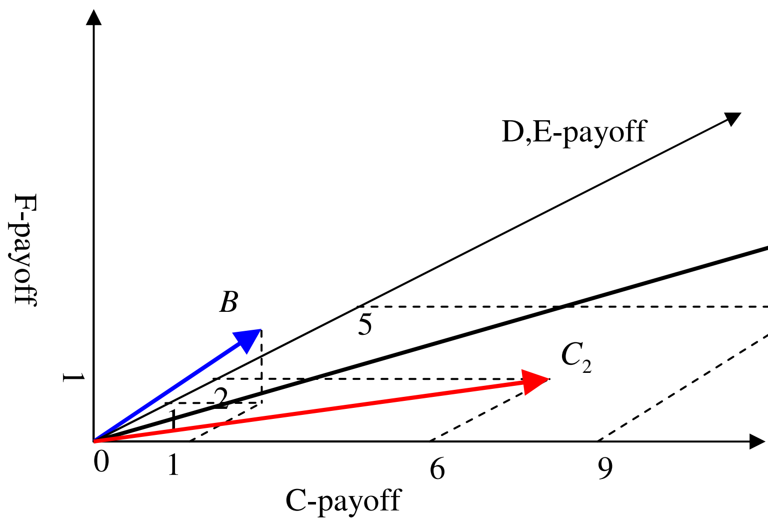
$\mathcal{F}_0 = \{\Omega, \varnothing\}$



$\mathcal{F}_1 = \{\{\omega_1, \omega_2\} \equiv A,$
$\{\omega_3, \omega_4\} \equiv B, \ldots\}$



$\mathcal{F}_2 = \{\{\omega_1\} \equiv C, \{\omega_2\} \equiv D,$
$\{\omega_3\} \equiv E, \{\omega_4\} \equiv F, \ldots\}$



and we can clearly see that C_2 does not belong to the hyperplane spanned by S_2 and B at time 2, therefore no static replication is possible, only the dynamic one through ϕ:

$$\begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} \notin \text{Span}\left\{ \begin{pmatrix} 9 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

6.5 Hedging ORA

6.5.1 Why to hedge ORA

A very good question indeed is what are exactly the reasons that an insurance company should hedge the ORA option. Note the difference between a financial institution which would only sell an ORA option without the ORA pension policy, such an institution would have to cover its naked option position in order to protect itself from the excessive risk. The case of the bank it different: it sells the whole product, and therefore is entitled to receive the premiums, which are then physically invested; therefore the position is only "partially" naked. In other words: it's "partially covered" by the compensation (loss-carry-forward mechanism) and the fact that the bank earns returns from its invested premiums.

Nevertheless the following relevant risks were identified, related to these two "partial covers":

- *premium investment returns risk*:
  - u-rate gets substantially bellow the guaranteed rate in a given year, so that that reserves are insufficient to guarantee the payoffs that year (in other words: there are not enough reserves to wait until the next year where the positive u-rate development and the "loss-carry-forward" mechanism would possibly make up for the loss,
  - u-rate gets under the guaranteed rate and "stays there", here we speak about the systematic decline in the u-rate over a longer period of time[1],
- *loss-carry-forward risk*:
  - product discontinuation, in this case there are simply no (sufficient) incoming premiums to cover the possible losses in the previous years.

6.5.2 Theoretical example how to hedge in incomplete markets/models

What could be done if we accept the notion of incomplete market for our hedging problem? In this case we can target for some sub-optimal dynamical hedge, which somehow approximates the payoff of our complex option. This problem is studied in [20], where the methods of the stochastic optimization/programming to find the solution to the following formal problem: given an arbitrary payoff function and a set of fundamental securities, find a self-financing dynamic portfolio strategy involving only the fundamental securities that most closely approximates the payoff in a mean-squared sense.

The crucial assumption in their approach is that there exists a finite-dimensional vector Z_t of state variables whose components are not perfectly correlated with the prices of any traded securities P_t, and $\begin{pmatrix} P_t \\ Z_t \end{pmatrix}$ is a vector Markov process. It is precisely the presence of this not perfectly correlated extra state Z_t, which makes this market model incomplete. An example of such state could be the stochastic volatility (2-factor Hull&White model) or stochastic autocorrelation of the ORA option.

[1] "The markets can stay down longer than you can stay solvent".

The method can be described as follows: with the self-financing (sub-optimal) portfolio defined as:

$$V_t = \theta_t P_t + B_t$$

with θ_t the hedging strategy and B_t the bond price, we define the replication error as:

$$\varepsilon(V_0) \equiv \sqrt{\min_{\{\theta_t\}} \mathbb{E}[V_T - F(P_T, Z_T)]^2}$$

and if we minimize this with respect to the initial wealth V_0:

$$\varepsilon^* \equiv \min_{\{V_0\}} \varepsilon(V_0)$$

we get something which we might intuitively call the "market-incompleteness-measure", which is thus zero for a complete markets/model (like Black and Scholes where the perfect replication is possible).

The choice of that non-perfectly correlated extra Markov state variable (the "source" of incompleteness), is dependent on the option we are modeling. For ORA (an Asian arithmetic-average-type option), the following state variable Z_t could be taken:

$$Z_{t_i} \equiv \frac{1}{i+1} \sum_{k=0}^{i} P_{t_k}$$

The solution of this stochastic optimization problem is a "classical" recursively defined sub-optimal replicating strategy θ, which is thus calculated numerically on a grid of possible values of P (and consequently Z). The recursive procedure involves several expectations of the functions of $\begin{pmatrix} P_t \\ Z_t \end{pmatrix}$ given $\begin{pmatrix} P_{t-1} \\ Z_{t-1} \end{pmatrix}$, thus in order to evaluate them we need to assume some probability distribution for P.

Nice thing about this so called $\varepsilon-$arbitrage is that in the Black-Scholes settings but with discrete trading it shows that even (continuous) Black-Scholes delta is less optimal than this $\varepsilon-$arbitrage hedging strategy. This kind of discrete-trading-market is also called *dynamically-incomplete.*

6.5.3 Theoretical possibility of the ORA hedge in (pretentiously) complete markets/models

If we content ourselves with a model that describes a complete market, we can see if our option payoff function satisfies the conditions of the *Martingale Representation Theorem* (see [5]), and hence could be represented by a stochastic integral (in the "world" driven by 1-dimensional Brownian Motion, actually a 1-factor model):

$$V(t) = \int_0^t \theta(s) dW_s$$

where $\theta(s)$ is a self-financing replication strategy, and thus also a dynamic hedge function.

The crucial condition here is the continuity of the payoff function, first we state the compensation function, see (4.3.2), in the continuous settings (with earnings E defined as in the chapter 4.3):

$$\begin{cases} C(t) = \left(\displaystyle\int_{g(C(s)|s \le t)}^{t} E(u)du \right)^{-} \\ g(C(s) \mid s \le t) = \max(w \mid w \le t, C(w) = 0, C(w+\varepsilon) > 0 \text{ for sufficiently small } \varepsilon) \end{cases}$$

and then the (continuous) payoff is:

$$V(t) = \left(E(t) - C(t)\right)^{+}$$

This (inconvenient) implicit payoff function definition is useless for any analytical purposes. It can merely serve as for proving that the payoff function is continuous, a proof we indeed do not state here.

In reality some more simple approximations/assumptions are used to price and hedge such arithmetic-average option, see [23] and [21] for more details, but basically it's not very different to the Levy's method, see 4.4, hence doing basically two things:

- approximating arithmetic average of the rates with a lognormal random variable,
- assuming some tractable and simple autocorrelation structure of the rates.

A further improvement in [21] is to use the geometric average of the same rates in order to estimate upper and lower bound for the arithmetic average option, and then adjust the strike price of the arithmetic option correspondingly. This complex loss-carry-forward mechanism of our ORA option could be probably modeled in the same way, by estimating the "average" compensation from the historical data, and then increase the option's strike price accordingly.

6.5.4 Hedging ORA and compensation mechanism

In fact the hedging and compensation problems for ORA are completely interconnected. The current situation is a little bit “peculiar”:

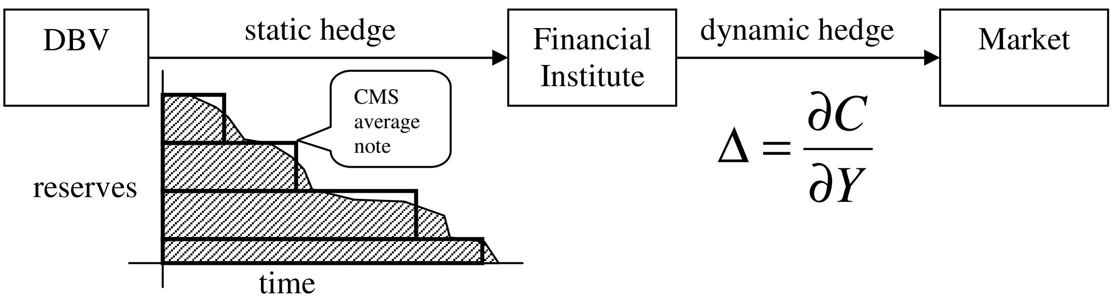
- SNS bank gives the ORA option to the customer free of charge,
- in other words SNS does not receive any money from the customers to set up a protective opposing hedge for this call option,
- and therefore it “channels through” the possible losses from this option to the customer via this compensation/loss-carry-forward mechanism.

Compare this with the standard situation where a financial institute sells the option to the customer and hedges it properly: there is no need for any kind of compensation mechanism, as the position is covered. In other words, if SNS bank sets up the hedge anyway (from its “own money”), the option could be priced as a standard call option, without taking the compensation in account. This would increase the price of the ORA option compared to the current situation, in fact resulting in giving even more valuable option to the customer at the end. This should be taken into account explicitly, and several connected issues like solvency (or other bank regulations) should be reconsidered.

Hence through the proper hedging we can get rid of this awkward (to model) compensation mechanism, and consider ORA a “standard” call option on this (still troublesome) arithmetic average of the swap rates. The only remaining problem is to model the autocorrelation of the rates, as the marginal distributions fit is the lesser problem, see 5.4. We try to tackle this problem(s) via the *Markov Functional* method, see [3] for more, described in the next chapter.

6.5.5 Practical hedge example from DBV

This example comes from the SNS-REAAL subsidiary DVB, see [22] for more detail:



The insurance company DBV buys a *static* hedge by a “Financial Institute”, in this case the hedge consists of the “custom-tailored” CMS average notes, where the averaging period is dependent on the individual part of the reserves that has to be covered. In the diagram we see 4 CMS average notes used as a hedge, each with different notional and averaging period. The Financial Institute has in turn used the *dynamical* hedge via the market to cover its own short position on these CSM notes (here a delta-hedging of this CMS average note C based on the current rate Y is displayed).

6.5.6 Practical hedge example form the Barclays Capital bank

This example is borrowed from the presentation/workshop given by Barclays Capital to SNS REAAL concerning specifically the hedging of the life insurance products like ORA, reference [18] contains more details.

One of their hedging strategy could be best described as a “semi-dynamic”/”semi-static” or ”static-dynamic” hedging. The strategy is to compose the replicating hedging portfolio (static aspect) in the way that it does not need to be rebalanced (dynamic aspect) very often. This is achieved via solving of the following optimization problem:

$$\min_{w} \sum_{i=1}^{N} w_i \Theta_i, \quad \text{subject to} \quad \begin{cases} \sum_{i=1}^{N} w_i \nu_i \le \nu \\ \sum_{i=1}^{N} w_i T_i \le T \\ \sum_{i=1}^{N} w_i C_i \le C \\ \sum_{i=1}^{N} w_i \Gamma_i \le \Gamma \end{cases}$$

where:

- w_i is the amount of the i-th replicating instrument that should be present in the replicating portfolio,
- Θ_i is the theta of the i-th replicating instrument,
- other indexed Greeks are standard,
- T_i is the i-th replicating instrument notional,
- C_i is the i-th replicating instrument costs,
- on the right side of the inequalities are the desired target values (e.g. Greeks and notional of the ORA portfolio we would like to hedge).

We see that we actually try to optimize on Θ, the time-decay of the portfolio, little bit contrary to our previous statement that Θ-hedging is useless, but here it will guarantee that the resulting replicating hedge will have little sensitivity to time: meaning that the hedge does not have to be rebalanced too frequently. This hedging strategy could be best described, in the geometrical language of the chapter 6.4, as a projection of the (too complex and impossible to replicate perfectly) ORA contingent claim on the space of the plain vanilla products, where we can see these optimized w_i-s as the weights of the projection. In this way we obtain the best possible hedge/projection, which is closest to the unrealizable perfect ORA hedge.

7 Markov-Functional Models

Markov-Functional interest rate models (further on MF models) were for the first time introduced by Hunt, Kennedy and Pelsser in 2000 (see [3]). The basic idea of such models is that the price of the zero-coupon/pure-discount bonds are monotonic functions (**functionals**) of some low-dimensional (mostly 1, maximally 2 dimensions) process which is **Markovian** in some convenient martingale measure. This is actually "the best of both worlds", meaning:

- it takes over efficiency of the *Short Rate Models* like in chapter 2.3, because we need to keep track of only this single Markov process, without all that difficult calibration and interest-rates-relations theory behind it,
- while it does not make any assumptions about the shape of the marginal distributions, as this is usually the case in these *Short Rate Models*, thus effectively allowing for any volatility smile, tackling our first problem from 5.4 indeed,
- still allowing the control of the autocorrelation of the rates as it is usually the case in these *Short Rate Models*, see the second problem from 5.4,
- and it uses the current prices of liquid marketed instruments like in the *Market Models* from chapter 3, without the high dimensionality coming along with these models, as we have a separate process (factor) for each LIBOR/Swap rate we model there.

In other words the 1-factor MF model reminds the 1-factor short rate model, with one single "driving" process, but without pretending to model the relation between short, long and forward rates through taking this single process to be a non-marketed/virtual instantaneous short rate. This assumption is one of the causes of all that known problems with the calibration and difficult valuation formulas. We can maybe say that the MF model is a pragmatic version of the pretentiously smart short rate models[1].

7.1 Basic Assumptions

From the *Market Models* we take over the tenor structure $T_1,...,T_{N+1}$, with the terminal discount bond $D_{T_{N+1}} \equiv D_{N+1}$ as a numeraire ($D_i(t)$ is thus the price at time- t of the discount bond with maturity- i). This numeraire defines a terminal measure $\mathbb{Q}^{N+1}$, under which we define a martingale, our driving Markov process:

(7.1.1) $$dx(t) = \tau(t)dW^{N+1}$$

where W^{N+1} is thus the Brownian Motion process under $\mathbb{Q}^{N+1}$, and $\tau(t)$ is the deterministic function to be determined later. In the *Short Rate Models* this driving process is indeed the instantaneous short rate ($x(t) \equiv r(t)$ there).

[1] The author can not help himself to make a parallel between a French hors-d'oeuvre and Dutch "broodje kaas" (bread with cheese). They do the same "trick", but the second one is comparably efficient and pragmatic while less pretentious.

For such process x we know the conditional distribution of $x(t)$ based on $x(s), s \le t$:

$$\phi(x(t) \mid x(s)) = \frac{\exp\left(-\frac{1}{2} \frac{(x(t)-x(s))^2}{\int_s^t \tau^2(u)du} \right)}{\sqrt{2\pi \int_s^t \tau^2(u)du}} \tag{7.1.2}$$

The functional form of the numeraire discount bond $D_{N+1}(t, x(t))$ is then determined by the prices of the liquid instruments, e.g. an interest rate cap (as a portfolio of caplets) or swaptions with different strikes. For such reference instrument $V_i^{ref}(K)$ maturing at i with strike K:

$$\frac{V_i^{ref}(K)}{D_{N+1}(0)} = \mathbb{E}^{N+1}\left[\frac{V_i^{ref}(D_{N+1}(T_i), K)}{D_{N+1}(T_i)} \right] = \int_{-\infty}^{\infty} \frac{V_i^{ref}(D_{N+1}(T_i, z), K)}{D_{N+1}(T_i, z)} \phi(z \mid x(0)) dz$$

a non-linear integral equation in D_{N+1}, which can be solved for D_{N+1}.

Because we work under the $\mathbb{Q}^{N+1}$, we can bring all the other (maturity-S) discount bonds under this measure and use their martingale characteristics there:

$$\frac{D_S(t)}{D_{N+1}(t)} = \mathbb{E}^{N+1}\left[\frac{D_S(S)}{D_{N+1}(S)} \middle| \mathcal{F}_t \right] = \int_{-\infty}^{\infty} \frac{1}{D_{N+1}(S, z)} \phi(z \mid x(t)) dz$$

and in this way we obtain all possible discount bonds functions for the whole tenor structure $T_1, ..., T_{N+1}$.

The idea is then to work on a grid of spatial (for process x) and time dimensions in order to fit the numeraire (T_{N+1}-maturity) discount bond to the prices of the liquid instruments, beginning at the terminal time T_{N+1} and working backwards towards T_1. Once the numeraire bond functional form is determined, all other discount bonds are determined too, and the price of an exotic instrument can be calculated relatively straightforwardly because of the assumed distribution of $x(t)$, actually a Gaussian one. The actual numerical implementation of the MF model was less trivial: it was plagued by the numerical instability, which was only solved with great difficulty and a lot of time. Another issue is the pricing of the exotic derivatives with very long maturities: to calibrate the MF model in this settings, the marketed instruments with a very few different strikes are available on the market. The sensitivity of the MF model to an inter-/extra-polation scheme for the different strikes is then too high. The authors of the original MF model also recognized that for the path-dependent exotic options the autocorrelation is much more crucial that the exact fit to the marginal distributions. Therefore a compromising solution was found: *Semi-Parametric Markov-Functional Model* (SMF), described in the following section.

7.2 Semi-Parametric Markov-Functional Model

Here we impose a functional form on the discount bonds, for the numeraire/terminal bond it looks like this:

(7.2.1)
$$\frac{1}{D_{N+1}(t)} = 1 + a_t e^{b_t x(t)} + d_t e^{-\frac{1}{2}c_t (x(t)-m_t)^2}$$

where we thus may choose (calibrate) the parameters $\{a_t, b_t, c_t, d_t, m_t\}$ for each time-instant from our tenor structure $T_1, ..., T_{N+1}$. Notice too that by taking $a_{N+1} \equiv 0, d_{N+1} \equiv 0$ we satisfy the "discount bond price terminal condition": $D_{N+1}(T_{N+1}) \equiv 1$.

Once we have the parameters for the numeraire bond for another time-instant $S : \{a_S, b_S, c_S, d_S, m_S\}$, we have the functional form for the discount bond with maturity- S too:

$$\frac{D_S(t)}{D_{N+1}(t)} = \mathbb{E}^{N+1}\left[\frac{1}{D_{N+1}(S)}\middle|\mathcal{F}_t\right] = \mathbb{E}^{N+1}\left[1 + a_S e^{b_S x(S)} + d_S e^{-\frac{1}{2}c_S (x(S)-m_S)^2}\middle| x(t)\right]$$

and if then $\left(x(S) \mid x(t)\right) \sim N\left(x(t), s^2(t,S)\right)$, we get an analytical expression:

(7.2.2)
$$\frac{D_S(t)}{D_{N+1}(t)} = 1 + a_{S,t} e^{b_S x(t)} + d_{S,t} e^{-\frac{1}{2}c_{S,t} (x(t)-m_S)^2}$$

where $a_{S,t} = a_S e^{\frac{1}{2}b_S^2 s^2}$, $d_{S,t} = \frac{d_S}{\sqrt{1+c_S s^2}}$ and $c_{S,t} = \frac{c_S}{1+c_S s^2}$.

The calibration then works as follows: we pick the market price $p(T,S,K)$ of the European put option with strike K, maturity T on the bond with maturity S, and we can state that under $\mathbb{Q}^{N+1}$ we have:

(7.2.3)
$$\frac{p(T,S,K)}{D_{N+1}(0)} = \mathbb{E}^{N+1}\left[\max\left\{0, \frac{K}{D_{N+1}(T)} - \frac{D_S(T)}{D_{N+1}(T)}\right\}\right]$$

which could be evaluated by using twice the following result, with N the standard normal cumulative distribution function:

(7.2.4)
$$\int_{-\infty}^{h}\left(1 + ae^{bx} + de^{-\frac{1}{2}c(x-m)^2}\right)\phi\left(\tfrac{h-\mu}{\sigma}\right)dx =$$
$$= N\left(\tfrac{h-\mu}{\sigma}\right) + ae^{b\mu+\frac{1}{2}b^2\sigma^2} N\left(\tfrac{h-(\mu-b\sigma^2)}{\sigma}\right) + \frac{d}{\sqrt{1+c\sigma^2}} e^{-\frac{1}{2}\frac{c(\mu-m)^2}{1+c\sigma^2}} N\left(\frac{h-\frac{\mu+mc\sigma^2}{1+c\sigma^2}}{\frac{\sigma}{\sqrt{1+c\sigma^2}}}\right) \equiv I(h)$$

as both terms in (7.2.3) have the same functional form of (7.2.1) with different parameters of course.

7.3 Autocorrelation and mean reversion

Let us not forget about the main reason we venture into these quite complex models to price and hedge our ORA product: the autocorrelation problem. Again in the analogy with the *Short Rate Models*, where the short rate process r follows the *Ornstein-Uhlenbeck* process (see [7] for more details):

$$dr = (\theta(t) - ar)dt + \sigma dW$$

we see that the autocorrelation of the spot interest rates depends on the mean-reversion parameter a:

$$Corr(r(t), r(s)) = \sqrt{\frac{e^{2at} - 1}{e^{2as} - 1}} \quad ,t < s$$

Now if we choose the diffusion parameter in our Markov driving process from (7.1.1) to be:

$$\tau(t) = e^{at}$$

the variance of the driving Markov process $x(s)$, given $x(t)$, would be:

$$Var(x(s) \mid x(t)) = \left\langle x_t + \int_t^s e^{au} dW_u \right\rangle = \int_t^s e^{2au} d\langle W \rangle_u = \int_t^s e^{2au} du = \left. \frac{e^{2au}}{2a} \right|_t^s = \frac{e^{2a(s-t)} - 1}{2a}$$

where of course $a > 0$, for $a = 0$ we get the Brownian Motion variance, therefore:

$$Var(x(s) \mid x(t)) = \begin{cases} \dfrac{e^{2a(s-t)} - 1}{2a}, & a > 0 \\ s - t \quad , & a = 0 \end{cases}$$

The covariance of $x(s)$ given $x(t)$ is for $a > 0$:

$$Cov(x(t), x(s)) = \mathbb{E}\left[x_t x_s\right] = \mathbb{E}\left[\int_0^t e^{av} dW_v \int_0^s e^{au} dW_u\right] = \frac{e^{2at} - 1}{2a} \quad ,t < s$$

therefore for the correlation for $a > 0$:

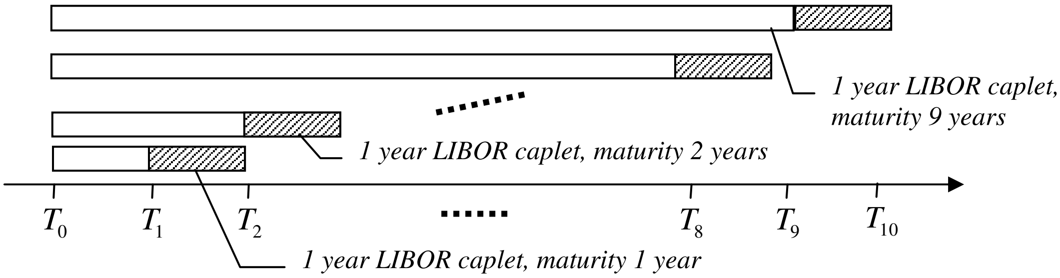
$$Corr(x(t), x(s)) = \frac{Cov(x(t), x(s))}{\sigma_t \sigma_s} = \frac{\frac{e^{2at} - 1}{2a}}{\sqrt{\frac{e^{2at} - 1}{2a}} \sqrt{\frac{e^{2as} - 1}{2a}}} = \sqrt{\frac{e^{2at} - 1}{e^{2as} - 1}} \quad ,t < s$$

and for $a = 0$ we have $Cov(x(t), x(s)) = t$ and $Corr(x(t), x(s)) = \sqrt{\frac{t}{s}}$.

We see that we have the same auto-correlation structure as the one with the mean-reversion. We would interpret then our parameter a as the "mean-reversion" of our MF model, gaining control of the autocorrelation through varying this parameter. Notice also that the discount bond functions are non-linear functions of x, and thus would not have exactly this autocorrelation form. And last but not least - as with all one-factor models, the cross-correlation between the individual bonds is exactly 1, nevertheless our ORA option is a path-dependant derivative, and by the way of using this MF model "mean-reversion" we gain some control over the autocorrelation of the swap rates in our ORA model.

7.4 Example of SMF using cap (portfolio of caplets)

This is an example of the SMF model coming from [3]. We will use the current market prices of the 1-year LIBOR caplets[1] to calibrate the discount bonds functions. Our tenor structure is $T_1,...,T_{10}$, hence we use D_{10} as numeraire and fit the prices of the caplets with the maturities of $1,2,...,9$ years respectively. The initial term structure is flat and given by $D_{T_n} = (1.05)^{-n}$. Let's state this graphically:



Now we have to "translate" the caplet into the corresponding option on the discount bond, let's take this 1-year LIBOR caplet with strike $\tilde{K}$ maturing at T_n, then its payoff is positive on the set:

$$\{\omega : L_{T_n}(\omega) > \tilde{K}\} \equiv \{L > \tilde{K}\}$$

and using the relation between 1-year LIBOR and 1-year discount bond (see [3]):

$$L_n(t) = \frac{D_n(t) - D_{n+1}(t)}{D_{n+1}(t)} = \frac{1}{D_{n+1}(n)} - 1 \equiv \frac{1}{D} - 1$$

Taking $t = n$, we can arrive at the following set equalities:

$$\{L > \tilde{K}\} = \{\tfrac{1}{D} - 1 > \tilde{K}\} = \{\tfrac{1}{D} > \tilde{K} + 1\} = \{\tfrac{1}{\tilde{K}+1} > D\} = \{K > D\}$$

with $K = \frac{1}{\tilde{K}+1}$.

Now we see that such caplet payoff is equivalent with the payoff of the put option with maturity T_n written on the discount bond maturing in 1 year. The only difference is the payoff time: the caplet, as an interest rate derivative, is paid off at the end of its tenor, T_{n+1} in this case, while the put option on bond is being paid off right at the option's maturity T_n. Therefore at time T_n the value C_n of the caplet's payoff at T_{n+1} must be the discounted payoff:

$$C_n(T_n, T_{n+1}, \tilde{K}) = \frac{1}{1+L}(L - \tilde{K})^+ = (1+\tilde{K}).\left(\frac{1}{1+\tilde{K}} - \frac{1}{1+L}\right)^+ = \left(1 - \frac{1+\tilde{K}}{1+L}\right)^+ = \left(1 - (1+\tilde{K}).D\right)^+$$

---

[1] For each tenor-time-instant we need at least 5 different strikes, as we try to calibrate 5 parameters for each time-instant, therefore needing at least 5 constraints for our optimisation problem, we should use more to avoid over-fitting.

With the caplet price $C(T_n, T_{n+1}, \tilde{K})$, thus caplet maturity T_n, written on annual LIBOR (period between $[T_n, T_{n+1}]$) and strike $\tilde{K}$, we then have:

$$C(T_n, T_{n+1}, \tilde{K}) = D_{n+1}(0).\mathbb{E}^{n+1}(L - \tilde{K})^+ = D_{n+1}(0).\mathbb{E}^{n+1}(1 - (1 + \tilde{K}).D)^+$$

working under the "original" caplet measure with the T_{n+1} – maturity discount bond as numeraire.

To use this market price in our SMF model we have to change the numeraire, see 3.1, finally getting:

$$\text{(7.4.1)} \qquad C(T_n, T_{n+1}, \tilde{K}) = D_{N+1}(0).\mathbb{E}^{N+1}\left[\max\left\{0, \frac{1}{D_{N+1}(T_n)} - (1 + \tilde{K}).\frac{D_{n+1}(T_n)}{D_{N+1}(T_n)}\right\}\right]$$

where the expectation could be of course calculated as easy as in (7.2.3).

We also need to note that because of (7.2.2), for time $t = 0$ we actually have (as $x(0) \equiv 0$):

$$\frac{D_S(0)}{D_{N+1}(0)} = 1 + a_{S,0}e^{b_S x(0)} + d_{S,0}e^{-\frac{1}{2}c_{S,0}(x(0)-m_S)^2} = 1 + a_S e^{\frac{1}{2}b_S s^2} + \frac{d_S}{\sqrt{1 + c_S s^2}} e^{\frac{-\frac{1}{2}c_S m_S^2}{1 + c_S s^2}}$$

and this implies for a_n:

$$\text{(7.4.2)} \qquad a_n = \frac{\dfrac{D_n(0)}{D_{N+1}(0)} - 1 - \dfrac{d_n}{\sqrt{1 + c_n s^2}} e^{\frac{-\frac{1}{2}c_n m_S^2}{1 + c_n s^2}}}{e^{\frac{1}{2}b_n s^2}}$$

hence at each time step the first parameter a_n is implied by the initial term-structure and the remaining four parameters, therefore we have only 4-dimensional optimisation problem at the end.

The following issues should be addressed later on:

- in this concrete example we use the "flat" volatility smile, this could pose some doubts on the applicability of the SMF where the volatility smile is substantial,
- the monotonicity in (7.2.1) must be preserved in each iteration step of the numerical method, we should pose an explicit constraint on $\{a_n, b_n, c_n, d_n, m_n\}$ addressing this,
- the same question exists about (7.2.3), where it is even more serious, as a difference of two monotonic functions is not necessarily monotone, we have to see if we can pose an explicit constraint on $\{a_n, b_n, c_n, d_n, m_n\}$ given $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}, m_{n+1}\}$.

7.4.1 Numerical method

Hence the algorithm would go as follows:

1) we would like to calibrate on 9 caplets, each maturing in 1,2,…,9 years, written on 1-year LIBOR rate, therefore we pick 9 different volatilities for the annual LIBOR rates, calculate at least 9x5 caplet prices (for each caplet at least 5 different strikes), through the Black's formula - using the given initial flat term structure, thus $L_i(0) = 0.05, \forall i$

2) start with the 9-year caplet, hence from (7.4.1), with i standing for different strikes:

$$c_i \equiv c(T_9, T_{10}, \tilde{K}_i) = D_{10}(0).\mathbb{E}^{10}\left(\tfrac{1}{D_{10}(T_9)} - (1+\tilde{K})\right)^+ = (1.05)^{-10}.\mathbb{E}^{10}\left(\tfrac{1}{D_{10}(T_9)} - (1+\tilde{K})\right)^+$$

3) make an initial guess for the parameters $\{b_9, c_9, d_9, m_9\}$, a_9 is implied by (7.4.2)

4) find the integration boundary h_i for $I(h_i)$ from (7.2.4) for each different strike i through some numerical method (let's call this the <u>inner</u> numerical method) to find for which $h_i = x^*$ we have ($K = \frac{1}{1+\tilde{K}}$ again):

$$\{x < x^*\} = \left\{1 + a_9 e^{b_9 x} + d_9 e^{-\frac{1}{2}c_9(x-m_9)^2} > \tfrac{1}{K}\right\} = \left\{\tfrac{K}{D_{10}(T_9)} > 1\right\}$$

where we know that such $h_i = x^*$ exists because of the monotonicity of (7.2.1)

5) with the integration boundaries h_i for (7.2.4) we can calculate the expectations in (7.4.1) and see how we should change the parameters $\{b_9, c_9, d_9, m_9\}$, a_9 is again implied by (7.4.2), in order to fit to the given caplet price. We optimize here on the square root of the sum of the quadratic errors, hence for the goodness-of-fit $\widehat{e}$:

$$\widehat{e} = \sqrt{\sum_{i=1}^{5} e_i^2} : e_i = c(T_9, T_{10}, \tilde{K}_i) - I(h_i) \qquad (7.4.1.1)$$

notice we use 5 strikes here, and let's call this the <u>outer</u> numerical method, as we iterate back to 4) from here until we have the satisfactory fit

6) with $\{a_9, b_9, c_9, d_9, m_9\}$ determined we go on with the next caplet: maturing in 8 years, again on 1-year LIBOR, and we get again from (7.4.1), omitting the different strikes i in the notation for simplicity:

$$c(T_8, T_9, \tilde{K}) = D_{10}(0).\mathbb{E}^{N+1}\left[\max\left\{0, \frac{1}{D_{10}(T_8)} - (1+\tilde{K}).\frac{D_9(T_8)}{D_{10}(T_8)}\right\}\right]$$

where from (7.2.2) we have:

$$\frac{D_9(T_8)}{D_{10}(T_8)} = 1 + a_{9,8}e^{b_9 x(8)} + d_{9,8}e^{-\frac{1}{2}c_{9,8}(x(8)-m_9)^2}$$

with $a_{9,8} = a_9 e^{\frac{1}{2}b_9^2 s^2}$, $d_{9,8} = \dfrac{d_9}{\sqrt{1+c_9 s^2}}$ and $c_{9,8} = \dfrac{c_9}{1+c_9 s^2}$

7) therefore to determine $\{a_8, b_8, c_8, d_8, m_8\}$ we use $\{a_9, b_9, c_9, d_9, m_9\}$ from the previous step and iterate until the price of this caplet $c(T_8, T_9, \tilde{K})$ also fits in a satisfactory way.

We thus actually repeat the whole procedure for each tenor-time-instant $T_1, ..., T_{10}$ to find the corresponding parameters $\{a_n, b_n, c_n, d_n, m_n\}, n \in \{1, 2, ..., 9\}$ for (7.2.1) going from the terminal time-instant backwards.

7.5 SMF monotonicity constraints

As already mentioned it is very important for the proper working of the numerical methods involved in SMF model that the discount bond functions (and their difference in the second case) are monotone. The monotonicity is used to ensure that the calibration procedure would have a unique result. In the further extension of the SMF model in order to price and hedge ORA option the monotonicity is even more important, as it will be used twice more:

- when simulating the Markovian process $x(t)$, we of course need 1-to-1 relation between a value of this process and a discount bond we would like to simulate,
- ORA would involve usage of swaptions for calibration, this means the decomposition of the put on the coupon bearing bond (equivalent with swaption) into a portfolio of the single discount bounds, here is the monotonicity of the discount bond function crucial.

Therefore let's look more seriously on this constraint, we begin by the observation that for a strictly positive function f :

$$f \text{ monotone} \Leftrightarrow \frac{1}{f} \text{ monotone}$$

therefore we can work directly with $f(x) = 1 + ae^{bx} + de^{-\frac{1}{2}c(x-m)^2}$, let's differentiate it[1]:

$$f'(x) = ae^{bx} - cd(x-m)e^{-\frac{1}{2}c(x-m)^2}$$

For monotonicity we would like to have this (strictly positive) function $f(x)$ have only positive derivative:

$$f'(x) > 0 \Leftrightarrow d < \frac{abe^{bx+\frac{1}{2}c(x-m)^2}}{c(x-m)} \equiv h(x)$$

It is clear that this is equivalent with:

$$d < \min_x(h)$$

so we have to differentiate $h(x)$ and find its minimum through setting $h'(x) = 0$.

The algebra involved is quite tedious, but at the end we get a 2nd degree polynomial in x with two roots, where the following one gives the minimum value of $h(x)$ (notice that we already implicitly use the constraints $a, b, c, d > 0$):

$$x^* = \tfrac{1}{2}\left(-\tfrac{b}{c} + \tfrac{\sqrt{b^2+4c}}{c} + 2m\right)$$

and filling this back into $h(x)$ we get our constraint:

$$d < \frac{ab\exp\left(\dfrac{\frac{1}{2} + b(\frac{1}{4}\sqrt{b^2+4c} + cm - \frac{1}{4}b)}{c}\right)}{\frac{1}{2}(\sqrt{b^2+4c} - b)}$$

[1] The author is indebted to his 2nd supervisor Andre Ran for pointing out the crucial steps in this method.

7.5.1 Second monotonicity constraint

Here we tackle the second monotonocity issue with the equation

(7.4.1), where we would thus like to ensure the monotonicity of:

$$f(x) = (1 + ae^{bx} + de^{-\frac{1}{2}c(x-m)^2}) - (1 + \hat{a}e^{\hat{b}x} + \hat{d}e^{-\frac{1}{2}\hat{c}(x-\hat{m})^2}) = ae^{bx} + de^{-\frac{1}{2}c(x-m)^2} - \hat{a}e^{\hat{b}x} - \hat{d}e^{-\frac{1}{2}\hat{c}(x-\hat{m})^2}$$

where $\{\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{m}\}$ are coming from the previous iteration of the numerical method and now we would like to estimate $\{a, b, c, d, m\}$, therefore we would like to pose a similar constraint for d as in the previous case, so let's differentiate $f(x)$:

$$f'(x) = ae^{bx} - cd(x-m)e^{-\frac{1}{2}c(x-m)^2} - \hat{a}e^{\hat{b}x} + \hat{c}\hat{d}(x-\hat{m})e^{-\frac{1}{2}\hat{c}(x-\hat{m})^2}$$

and let's express $f'(x) > 0$ for d again:

$$d < \frac{ae^{bx} - \hat{a}e^{\hat{b}x} + \hat{c}\hat{d}(x-\hat{m})e^{-\frac{1}{2}\hat{c}(x-\hat{m})^2}}{c(x-m)e^{-\frac{1}{2}c(x-m)^2}} \equiv h(x)$$

This expression is much more complex that in the previous case, and unfortunately the minimum of $h(x)$, thus the solutions of $\{x^* : h'(x^*) = 0\}$ would have to be calculated numerically too, and then inserted back to $h(x)$ in order to see which value is the minimum. This extra complexity we would build in only in case that the monotonicity in

(7.4.1) would be a problem.

7.6 SMF and Monte Carlo method

The authors of the original MF model also tried to price the path dependant derivatives by incorporating an extra state variable for each point in the grid (e.g. number of caplets left in the *Chooser Cap*, see [3] for more detail). For the arithmetic average product like ORA option this would get too complicated, and moreover this would tackle only the pricing, not hedging problem. This led us, completely in the analogy with the *Short Rate Models*, to incorporate the Monte Carlo (further on MC) method[1] in our calibrated SMF model.

The idea is to calibrate the functions of the discount bond to the relevant instruments (caps/swaptions) and then to simulate our driving Markovian process $x(t)$ thorugh the MC method. We can validate this approach through "back-pricing" of the instruments we have used to calibrate the SMF model, thus we compare our SMF-MC prices with the original market prices of these calibrating instruments.

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[1] This is probably more of "un marriage de raison" than "un marriage de passion".

8 ORA pricing with SMF and Monte Carlo

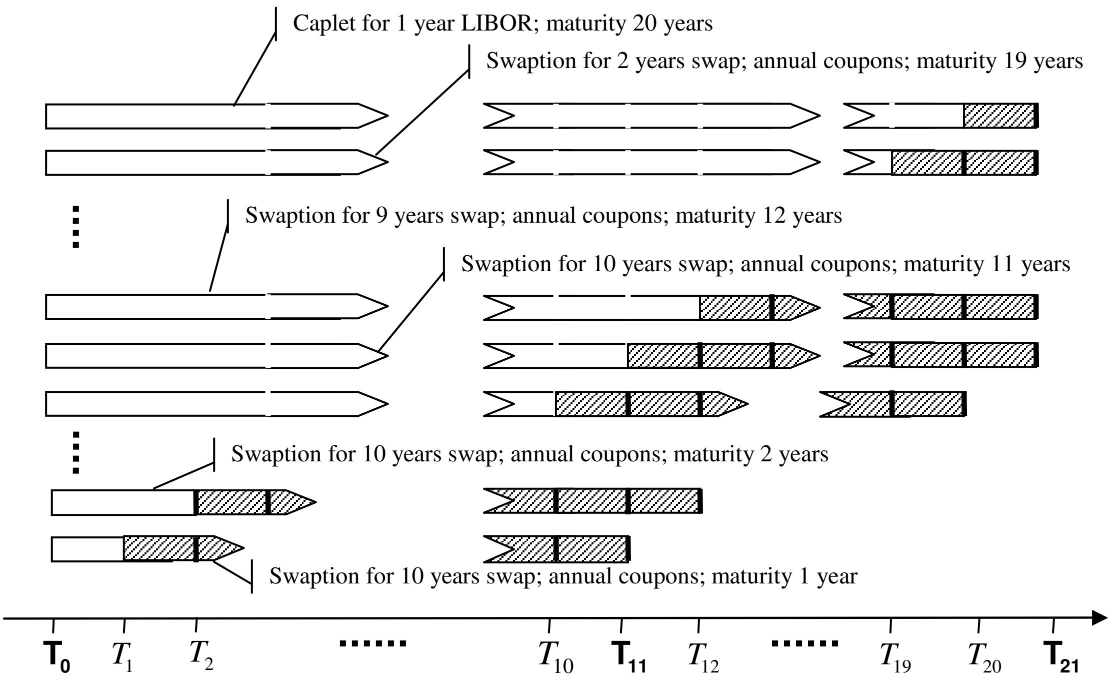
8.1 Model setup

Recall that the payoff of ORA (without loss-carry-forward/compensation mechanism) is:

$$V_{nocomp}(t) = \left(\sum_{i=t-10}^{t} (y_i - k) T_i(t) \right)^+ = \left(\sum_{i=t-10}^{t} (y_i - k) \right)^+ = \left(\left[\sum_{i=t-10}^{t} y_i \right] - \widehat{k} \right)^+ \qquad (8.1.1)$$

with (without loss of generality) for all tranches $T_i(t) \equiv 1, \forall i$ and $\widehat{k} = 10k$, and where y_i is the 10 year par swap rate in the year i (in some fixed month of the year i).

It seems clear that we would like to calibrate our SMF model for ORA through swaptions, let's suppose we want to calculate the price of ORA payoff taking place 11 years from now, meaning that we use 21th year as the terminal date, notice that we will divert from the notation of the previous sections by taking $N = 21$ (not $N + 1 \equiv 21$) in order to keep our previous notation from the swaps chapter, see 3.4. Therefore our tenor structure is $T_1, ..., T_{11}, ..., T_{21}$, where the first swap rate y_1 from (8.1.1) is thus the par swap rate on the payer swap for the period of $[T_1, T_{11}]$ with annual coupons, T_{11} is our payoff time and $D_{21} \equiv D_N$ is our numeraire. The initial term structure will be read off the market at T_0 which is the "current" time. For the calibration of the model we will use 20 payer swaptions (at least 5 different strikes each), maturing in $1, 2, ..., 20$ years, first 11 written on the 10 years swaps, last 9 written on swaps with maturity equal to the terminal date T_{21}, the last one is thus just a caplet on the annual LIBOR maturing in T_{20}. Let's draw this:



Let's "translate" the swaption into the corresponding portfolio of the put options on the discount bond, this is indeed the *Jamshidian decomposition* (ref [14]), we take such N-years swaption with strike $\tilde{K}$ maturing at T_n, its payoff is positive on the set:

$$\{\omega : y_{n,N}(T_n,\omega) > \tilde{k}\} \equiv \{y > \tilde{k}\}$$

Then using the relation between swap rates and discount bonds (see [3]):

$$y_{n,N}(t) = \frac{D_n(t) - D_N(t)}{\sum_{i=n+1}^{N} \alpha_i D_i(t)} = \frac{1 - D_N(T_n)}{\sum_{i=n+1}^{N} D_i(T_n)}$$

as we take $t = T_n$, and accrual factor $\alpha_i \equiv 1, \forall i$.

We also see that the following equalities of sets hold:

$$\{y > \tilde{k}\} = \left\{1 - D_N(T_n) > \sum_{i=n+1}^{N} \tilde{k} D_i(T_n)\right\} = \left\{1 > \sum_{i=n+1}^{N} c_i D_i(T_n)\right\} = \{1 > B_{n,N}(T_n)\}$$

where:

- $c_i = \begin{cases} \tilde{k}, & i = n+1, \ldots N-1 \\ 1+\tilde{k}, & i = N \end{cases}$,
- hence we see that the sum $\sum_{i=n+1}^{N} c_i D_i(T_n)$ is nothing else then just a coupon $(\tilde{k})$ bearing bond with face value 1, so we denote this sum with $B_{n,N}$.

What the set equalities again do not show is the difference in the payoff time of the put on the coupon bearing bond (right at the option's maturity) and the swaption (stream of payoffs at each tenor period), therefore in order to express the swaption's value at T_n we have to discount these individual "swaplets" payoffs back to T_n, getting:

$$\text{(8.1.2)} \quad PS_{n,N}(T_n) = \sum_{i=n+1}^{N} D_i(T_n).(y_{n,N}(T_n) - \tilde{k})^+ = P_{n+1,N}(T_n).(y_{n,N}(T_n) - \tilde{k})^+ = (1 - B_{n,N}(T_n))^+$$

Now as all discount bonds function are monotonic in x, we know that ("$\exists!$" means "exists unique"):

$$\text{(8.1.3)} \qquad \exists! x^* : \sum_{i=n+1}^{N} c_i D_i(T_n, x^*) = 1$$

Let's insert the sum in (8.1.3) into (8.1.2), using the functional form for the discount bonds, and then RHS of (8.1.2) becomes:

$$\left(\sum_{i=n+1}^{N} c_i \frac{1}{1 + a_i e^{b_i x^*} + d_i e^{-\frac{1}{2} c_i (x^* - m_i)^2}} - \sum_{i=n+1}^{N} c_i D_i(T_n) \right)^+ = \left(\sum_{i=n+1}^{N} c_i \left[\frac{1}{1 + a_i e^{b_i x^*} + d_i e^{-\frac{1}{2} c_i (x^* - m_i)^2}} - D_i(T_n) \right] \right)^+$$

Then because of the monotonicity of all discount bonds functions the previous is equivalent to:

$$\left(\sum_{i=n+1}^{N} c_i\left[\frac{1}{1+a_i e^{b_i x^*}+d_i e^{-\frac{1}{2}c_i(x^*-m_i)^2}}-D_i(T_n)\right]\right)^+ \sim \sum_{i=n+1}^{N} c_i\left(\left[\frac{1}{1+a_i e^{b_i x^*}+d_i e^{-\frac{1}{2}c_i(x^*-m_i)^2}}-D_i(T_n)\right]\right)^+$$

Hence we can decompose (8.1.2) into:

$$PS_{n,N}(T_n)=(1-B_{n,N}(T_n))^+ = \sum_{i=n+1}^{N} c_i(K_i-D_i(T_n))^+$$

where c_i is defined as before, and for the strikes:

$$K_i = \frac{1}{1+a_i e^{b_i x^*}+d_i e^{-\frac{1}{2}c_i(x^*-m_i)^2}}$$

Therefore the swaption (resp. put on the coupon bearing bond) pricing formula:

(8.1.4) $$PS_{n,N}(0)=P_{n+1,N}(0)\mathbb{E}^{n+1,N}(y_{n,N}(T_n)-\tilde{k})^+ = P_{n+1,N}(0)\mathbb{E}^{n+1,N}\frac{(1-B_{n,N}(T_n))^+}{P_{n+1,N}(T_n)}$$

could be decomposed into a portfolio of the puts on the single discount bonds:

$$PS_{n,N}(0)=\sum_{i=n+1}^{N} c_i P_{n+1,N}(0)\mathbb{E}^{n+1,N}\frac{(K_i-D_i(T_n))^+}{P_{n+1,N}(T_n)}$$

Let's summarize what we have got: we can decompose the payer swaption into the portfolio of the puts on the discount bonds:

$$PS_{n,N}(0)=P_{n+1,N}(0)\mathbb{E}^{n+1,N}(y-\tilde{k})^+ = \sum_{i=n+1}^{N} c_i P_{n+1,N}(0)\mathbb{E}^{n+1,N}\frac{(K_i-D_i(T_n))^+}{P_{n+1,N}(T_n)}$$

Now to be able to calibrate our SMF model to the current swaption market prices $PS_{n,N}(0)$ we have to change the numeraire, see 3.1, finally getting:

(8.1.5) $$PS_{n,N}(0)=\sum_{i=n+1}^{N} c_i D_N(0)\mathbb{E}^{N}\left(\frac{K_i}{D_N(T_n)}-\frac{D_i(T_n)}{D_N(T_n)}\right)^+$$

where the expectation could be of course calculated as easy as in (7.2.3), notice again that here we use T_N as a terminal date (and D_N numeraire) as opposed to (7.2.3) where T_{N+1} resp. D_{N+1} are used.

8.2 Calibration

For notational simplicity we omit the different strikes for one tenor-time-instant swaption here, as the incorporation of these is as straightforward as in chapter 0, using the same error measure for the goodness-of-the-fit.

So let's sketch the algorithm to estimate all the discount bond function parameters for all our tenor-time-instants $T_1,...,T_{10},...,T_{21}$, we begin from the end and take the last swap rate y_{20} which is equal to annual LIBOR in year 20, therefore we can use the caplet formula from (7.4.1):

$$C(T_{20},T_{21},\tilde{K}) = D_{21}(0).\mathbb{E}^{21}\left(\tfrac{K}{D_{21}(T_{20})}-1\right)^{+}$$

where of course from the caption's strike $\tilde{K}$ we have $K=\frac{1}{1+\tilde{K}}$.

We go on like in the caption's example from chapter 7.3, thus determining the parameters $\{a_{20},b_{20},c_{20},d_{20},m_{20}\}$ in this first iteration.

Second step is more difficult, here we have to use the swaption maturing in 19 years written on the 2 years swap, so using (8.1.5) we have:

$$PS_{19,21}(0) = \sum_{i=20}^{21} c_i D_{21}(0)\mathbb{E}^{21}\left(\frac{K_i}{D_{21}(T_{19})}-\frac{D_i(T_{19})}{D_{21}(T_{19})}\right)^{+} \tag{8.2.1}$$

and this is equal to:

$$D_{21}(0).\left[\tilde{k}.\mathbb{E}^{21}\left(\frac{K_{20}}{D_{21}(T_{19})}-\frac{D_{20}(T_{19})}{D_{21}(T_{19})}\right)^{+} + (1+\tilde{k}).\mathbb{E}^{21}\left(\frac{K_{21}}{D_{21}(T_{19})}-1\right)^{+}\right]$$

where the Jamshidian decomposition (identification of K_{20} and K_{21}) was performed based on (8.1.2) and the initial guess for $\{a_{19},b_{19},c_{19},d_{19},m_{19}\}$, given $\{a_{20},b_{20},c_{20},d_{20},m_{20}\}$, thus finding x^* such that:

$$1=\sum_{i=20}^{21} c_i D_i(T_{19},x^*) = \tilde{k}.D_{20}(T_{19},x^*) + (1+\tilde{k}).D_{21}(T_{19},x^*) = D_{21}(T_{19},x^*).\left(\tilde{k}.\frac{D_{20}(T_{19},x^*)}{D_{21}(T_{19},x^*)}+(1+\tilde{k})\right)$$

where $D_{21}(T_{19},x^*)$ is determined by $\{a_{19},b_{19},c_{19},d_{19},m_{19}\}$ and $\frac{D_{20}(T_{19},x^*)}{D_{21}(T_{19},x^*)}$ by $\{a_{20},b_{20},c_{20},d_{20},m_{20}\}$ through equation (7.2.2).

Now we can handle the two expectations in (8.2.1) exactly as before, thus (twice) finding some $h = x^*$ for which the term in the expectation is positive, and using result (7.2.4) to calculate the expectation. Summing up these two expectations gives the numerical method the goodness-of-fit error and (hopefully) steers the search for $\{a_{19}, b_{19}, c_{19}, d_{19}, m_{19}\}$ in the good direction.

Suppose now that we have just determined $\{a_{11}, b_{11}, c_{11}, d_{11}, m_{11}\}$ and with our initial guess for $\{a_{10}, b_{10}, c_{10}, d_{10}, m_{10}\}$ we are "switching" now to swaptions on the 10-years swaps, thus the underlying swaps do not have the terminal date maturity from this step on. This is desirable for two reasons:

- ORA is based on the 10-years swap rate,
- we have less calculations this way.

So let's see about this swaption, again using (8.1.5):

$$PS_{10,21}(0) = \sum_{i=11}^{21} c_i D_{21}(0) \mathbb{E}^{21}\left(\frac{K_i}{D_{21}(T_{10})} - \frac{D_i(T_{10})}{D_{21}(T_{10})} \right)^+$$

and this is:

$$D_{21}(0).\left(\tilde{k}.\mathbb{E}^{21}\left(\frac{K_{11}}{D_{21}(T_{10})} - \frac{D_{11}(T_{10})}{D_{21}(T_{10})} \right)^+ + \ldots + (1+\tilde{k}).\mathbb{E}^{21}\left(\frac{K_{20}}{D_{21}(T_{10})} - \frac{D_{20}(T_{10})}{D_{21}(T_{10})} \right)^+ \right)$$

where the Jamshidian decomposition goes as previously, and where thus $D_{21}(T_{10}, x^*)$ is determined by $\{a_{10}, b_{10}, c_{10}, d_{10}, m_{10}\}$ and $\dfrac{D_i(T_{10}, x^*)}{D_{21}(T_{10}, x^*)}, i = 11, \ldots, 20$ by $\{a_i, b_i, c_i, d_i, m_i\}$ through equation (7.2.2), already known parameters at this stage.

8.3 Numerical results

This is actually the main section of whole thesis; here we present the results of the valuation of the ORA option through the SMF-MC method. The valuation consists of two main steps:

- *model calibration*: we calibrate model based on the given initial term structure of the interest rates and the Black's prices of some plain vanilla instrument,
- *simulation*: we simulate the driving Markov process x in order to estimate the price of the ORA option, which is in fact a function of this process.

8.3.1 Calibration

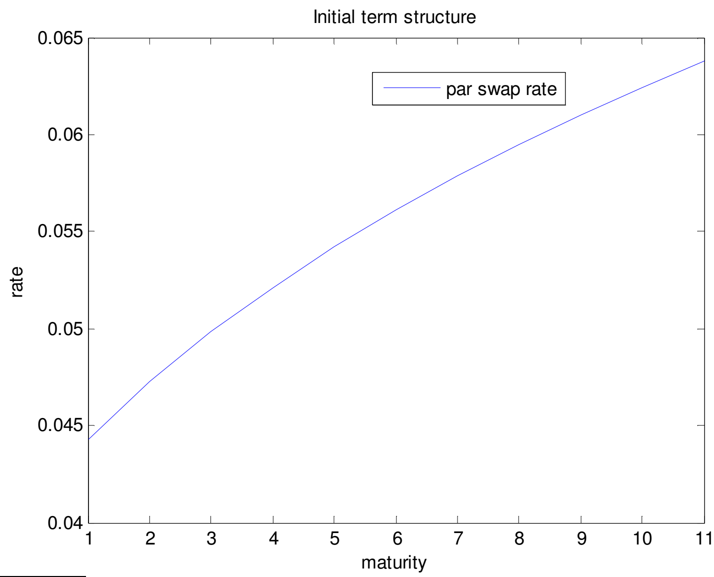
It might be a little bit of an overkill to begin straight away with the numerical procedure including the Jamshidian decomposition as described in 8.1. We can as well "cheat" a bit and use the numerical integration when calculating the expectation, thus omitting the Jamshidian decomposition, and directly bringing this expectation under the model martingale (N -forward) measure:

(8.3.1.1) $$PS_{n,N}(0) = P_{n+1,N}(0)\mathbb{E}^{n+1,N}\frac{(1-B_{n,N}(T_n))^+}{P_{n+1,N}(T_n)} = D_N(0)\mathbb{E}^N\left(\frac{1}{D_N(T_n)} - \sum_{i=n+1}^{N} c_i \frac{D_i(T_n)}{D_N(T_n)}\right)^+$$

where $c_i = \begin{cases} \tilde{k}, & i = n+1,\ldots N-1 \\ 1+\tilde{k}, & i = N \end{cases}$.

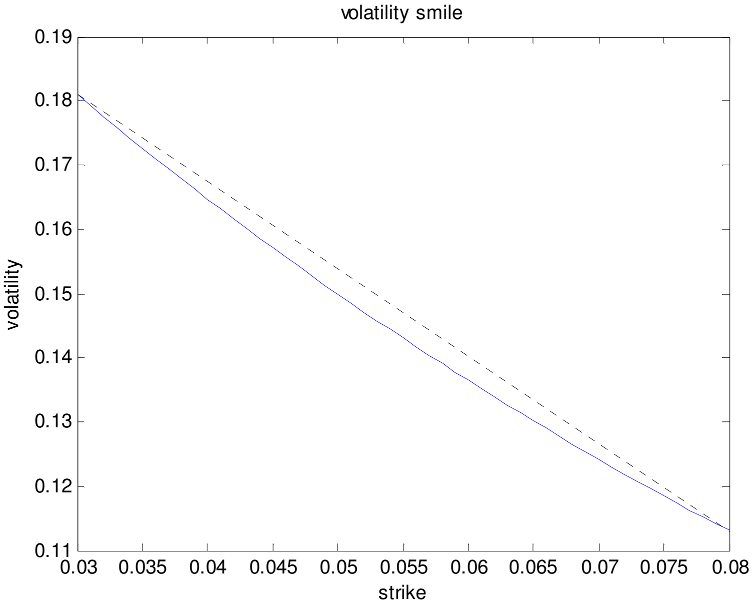
In order to evaluate this integral in (8.3.1.1) numerically we will use the *Adaptive Simpson Rule* numerical integration scheme, see [15] for more detail, on the interval of $[-5\sigma, +5\sigma]$, because we are dealing with the normally distributed random variable in our integral/expectation.

The initial term structure (par swap rates 1x10[1] up to 11x10) in our model is not flat in order to have more realistic settings:



[1] In this notation 2x10 rate means the par swap rate for 10 years swap starting in year 2.

We also use the (skewed) volatility smile, as the dotted straight line shows:



For each tenor-time-instant we have calculated the Black's price of the swaption (resp. caplet for the first one) according to the above volatility smile and initial term structure. We have used 10 strikes per instrument, to simulate the market conditions for valuation of the long-term maturity products like ORA, where just a couple of the strikes are available on the market for these long maturities. We have also used only one caption and couple of receiver swaptions, as opposed to the original caplet example in ref [3] where also floorlets were used. Thus the number of the calibrating instruments in our model was 10 per time-instant as opposed to 2x115 in ref [3], still giving the same acceptable result, which shows that SMF is quite robust.

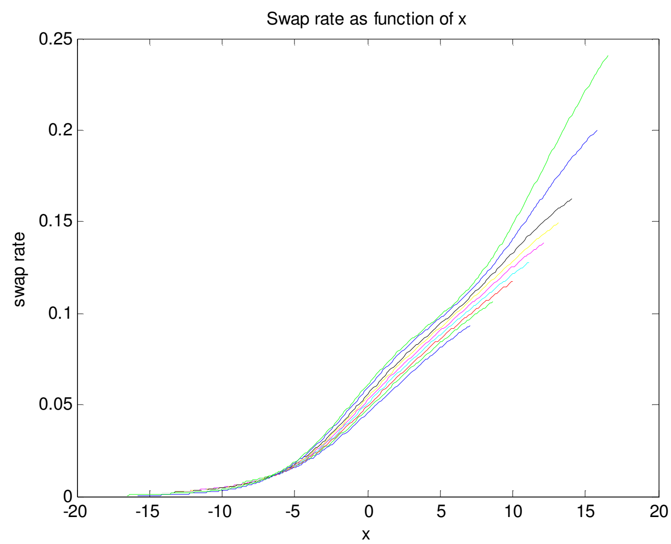
The calibration procedure itself was quite troublesome, but most troubles could be accounted to the optimization method used in connection with the numerical integral for (8.3.1.1). The monotonicity of the functional form (7.2.1), critical for the SMF model, was not maintained through the individual iterations. Moreover even the weaker condition of the positivity of a, b, c parameters was not sustained during the optimization. Only the resulting parameters satisfy these conditions. More proper implementation of the SMF model would require much more careful optimization method, the one which restricts the search for the optimal parameters only to the subspace where $a, b, c > 0$ and the monotonicity is guaranteed. Another huge issue was the mean-reversion: when this one was increased from 0% to 1% or more in one step, the whole (shaky) optimization fell apart. We had to "sneak up" on the higher mean-reversion parameters through using the optimal parameters form the zero mean-reversion as the initial values for the next step, and then gradually increasing the mean-reversion with 0.1% at a time, always using the last parameters as the initial values for the next run - until we had the parameters for mean-reversion in the range of 0% to 4%. Above 4% even this sneaky procedure did not provide us with usable parameters. This is no surprise, as we have actually told the optimizer to stay in the neighborhood of the previous solution, and thus not to search for the real global optimal parameters. Another suspected improvement could be achieved by using of the original SMF method, thus no numerical integration, but the Jamshidian decomposition as described in 8.1, followed by an analytical (exact) integration.

Nevertheless for mean-reversion between 0% and 4% we achieved quite satisfactory fit, as cold be seen from the following table for the mean reversion 0%, for each time-instant of our model we have 5 parameters:

	1	2	3	4	5	6	7	8	9	10
a	1.349	1.222	1.491	1.301	0.074	0.926	0.768	0.681	0.604	0.440
b	0.042	0.037	0.204	0.204	0.752	0.152	0.213	0.210	0.206	0.093
c	0.019	0.028	0.315	0.236	0.170	0.817	0.094	0.102	0.112	0.293
d	0.564	0.618	0.200	0.300	1.655	1.021	0.567	0.549	0.525	1.018
m	0.000	0.000	0.000	0.246	0.000	0.111	1.802	1.641	1.459	0.074
Error	0.286	0.486	0.030	0.016	1.088	0.458	0.002	0.002	0.003	0.990
Max fit error	0.165	0.289	0.012	0.007	0.617	0.153	0.001	0.001	0.001	0.343
	11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
a	0.396	0.354	0.035	0.030	0.236	0.124	0.125	0.080	0.050	0.050
b	0.218	0.210	0.487	0.465	0.188	0.175	0.198	0.205	0.202	0.123
c	0.088	0.098	0.171	0.156	0.101	0.237	0.074	0.067	0.065	0.833
d	0.532	0.470	0.901	0.776	0.279	0.502	0.200	0.162	0.109	0.049
m	1.763	1.546	0.260	0.401	1.528	0.438	1.698	1.737	1.738	1.003
Error	0.002	0.002	0.010	0.011	0.010	0.052	0.008	0.005	0.003	0.087
Max fit error	0.001	0.001	0.003	0.006	0.004	0.021	0.003	0.002	0.001	0.047

Here **Error** stands for root of the squared calibration error as defined in (7.4.1.1), and **Max fit error** is the maximum error between a calibration instrument and its SMF price with the parameters. We see that the overall fit is quite good, nevertheless e.g. by the time-instant 10 is the fit error 0.343, and we see that here the d parameter is far too high. Thus this is clearly a calibration error.

Let's draw now the swap rates as the functions of the driving process x, each one on the range of $[-5\sigma_T, +5\sigma_T]$ where σ_T is the standard deviation of the x process at the time T, where T is thus the maturity of the swap rate in the consideration (2x10 up to 11x10 rates):

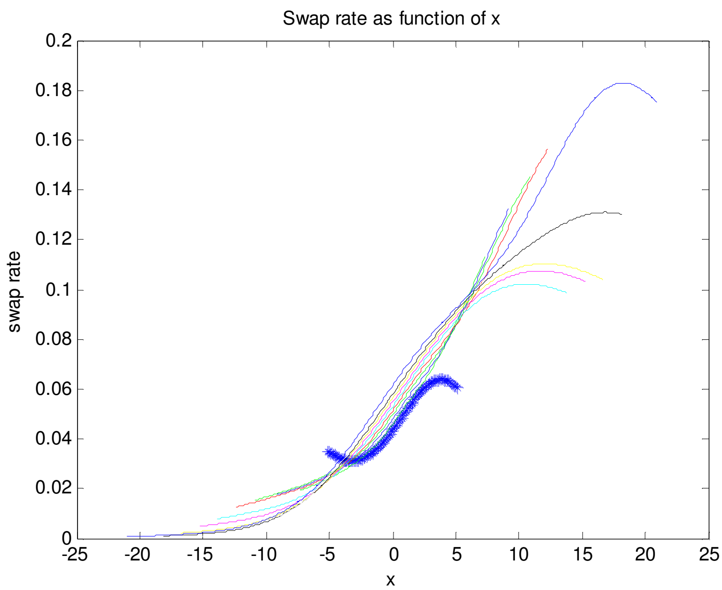


We see that the monotonicity is no problem on the interval where the swap rate has almost 100% probability, nevertheless mainly with the big Monte Carlo simulation we can have (and had) situations where a very low probability x was drawn, and the resulting swap rate was negative, because its function was not monotone beyond this interval. To counter this, we use in this case the average rate from that run instead of this negative value.

Now the same for mean-reversion of 4%:

	1	2	3	4	5	6	7	8	9	10
a	0.847	1.735	1.594	1.446	1.250	0.782	0.632	0.445	0.362	0.332
b	0.000	0.177	0.169	0.162	0.156	0.141	0.147	0.159	0.173	0.178
c	0.058	0.747	0.373	0.226	0.112	0.032	0.028	0.024	0.025	0.030
d	3.011	0.061	0.105	0.159	0.290	1.126	1.101	1.182	1.028	0.832
m	6.079	1.171	1.562	1.955	2.733	6.717	6.592	6.770	5.874	4.742
Error	0.307	0.091	0.041	0.022	0.012	0.037	0.027	0.021	0.012	0.003
Max fit error	0.202	0.055	0.020	0.010	0.007	0.019	0.014	0.011	0.006	0.001
	11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
a	0.312	0.168	0.054	0.027	0.055	0.106	0.088	0.056	0.041	0.049
b	0.176	0.200	0.268	0.301	0.228	0.163	0.153	0.155	0.138	0.075
c	0.033	0.026	0.022	0.021	0.022	0.027	0.027	0.024	0.024	0.298
d	0.683	0.770	0.832	0.745	0.537	0.333	0.253	0.196	0.123	0.056
m	4.094	4.823	5.542	5.675	4.764	3.555	3.321	3.397	3.308	1.820
Error	0.006	0.004	0.004	0.008	0.004	0.005	0.005	0.004	0.003	0.086
Max fit error	0.003	0.001	0.002	0.003	0.002	0.002	0.002	0.002	0.001	0.045

For this higher mean reversion the things have gone seriously wrong for time-instant 1, we see that d is quite huge there, let's see the impact on the monotonicity on the relevant rates[1] (we draw 1x10 rate too now):



We see immediately that for the 1x10 rate (the thick "S") there is no real monotonicity, luckily this rate in not part of the ORA payoff. Nevertheless it was time to stop with such shaky calibration when mean-reversion 4% was reached. We are confident that with the proper calibration method any mean-reversion can be reached.

[1] Rates involved in ORA payoff computation.

To really see the quality of the calibration in more detail, we plotted the 5x10 swap rate distribution implied by the SMF model parameters, as compared to the distribution implied by the Black's price of the calibrating swaption on the same rate. The SMF distribution could be calculated as follows, with f_Y the probability density function of the swap rate Y , Y itself a function of X , where X is the "snapshot" of our driving process x at the swap rate maturity T , thus we work under the probability measure $\mathbb{P}$ of $X \sim N(0,\sigma_T^2)$:

$$f_Y(y) = \frac{dF_Y}{dy}(y)$$

and then with $g : Y = g(X)$, as each swap rate is a function of the discount bonds and they in turn have the SMF functional form of (7.2.1):

$$F_Y(y) = \mathbb{P}(Y < y) = \mathbb{P}(g(X) < y) = \mathbb{P}(X < g^{-1}(y)) = F(g^{-1}(y))$$

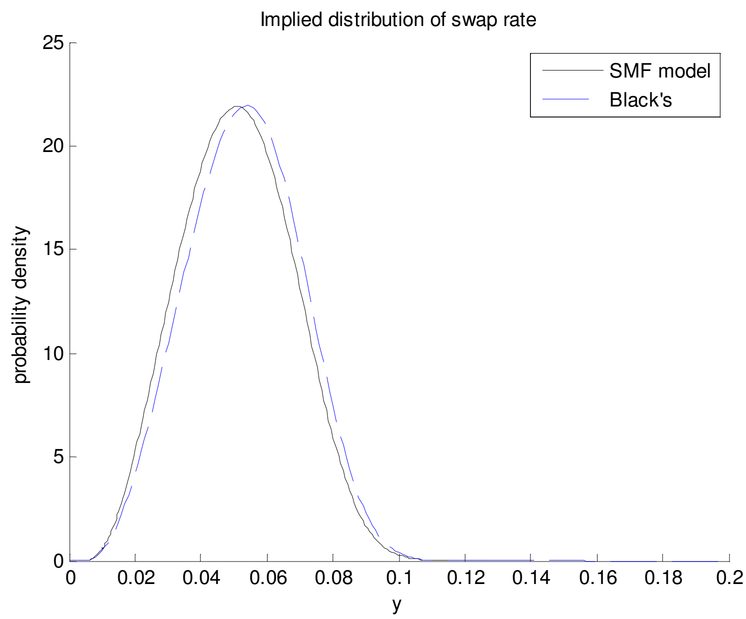
where $F \equiv F_X$, plus we have used the fact that g is the monotone function of X , and then:

$$\frac{dF_Y}{dy}(y) = \frac{dF(x)}{dg(x)} : x = (g^{-1}(y))$$

and then with the shorthand notation:

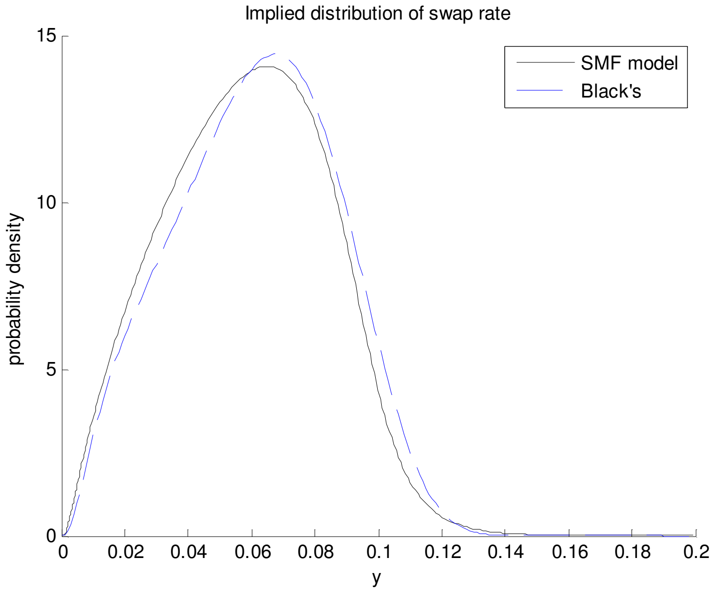
$$\frac{dF(x)}{dg(x)} = \frac{\frac{dF(x)}{dx}}{\frac{dg(x)}{dx}} = \frac{f(x)}{g'(x)}.$$

Now as the distribution of X at T together with g are known, we can plot the 5x10 rate distribution, the Black's distribution is plotted using (5.3.1):

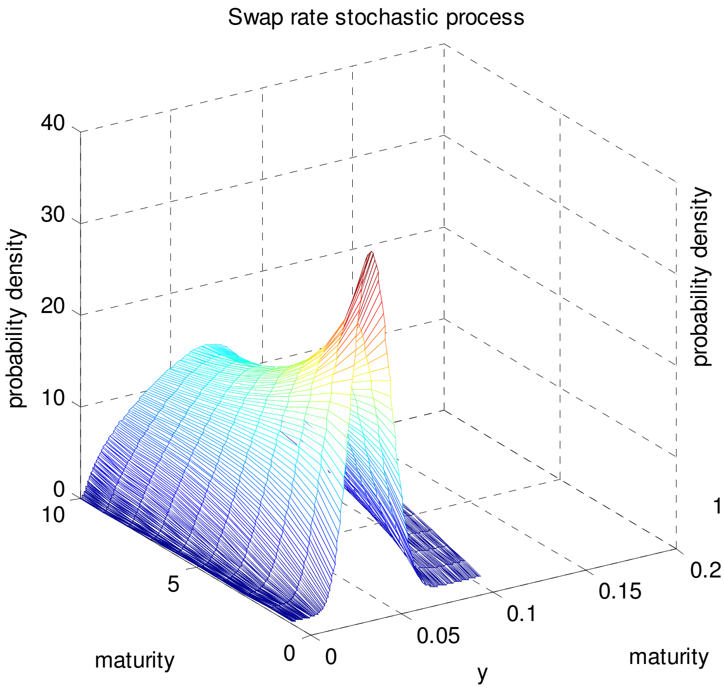


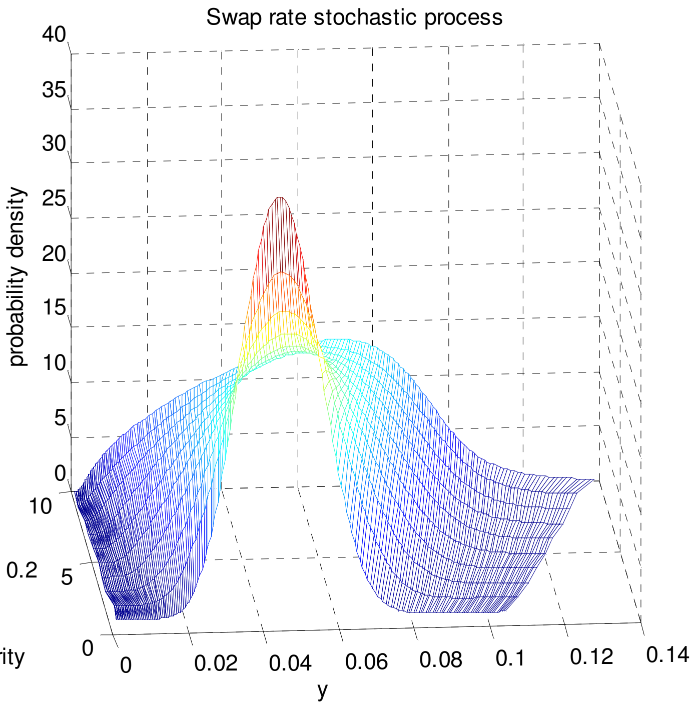
We see that the fit is surprisingly fine, given that we are using the parameters with the mean-reversion level of 4%. The other rates show similar satisfactory fit.

This is the same diagram, now for 10x10 rate and mean-reversion 0%:



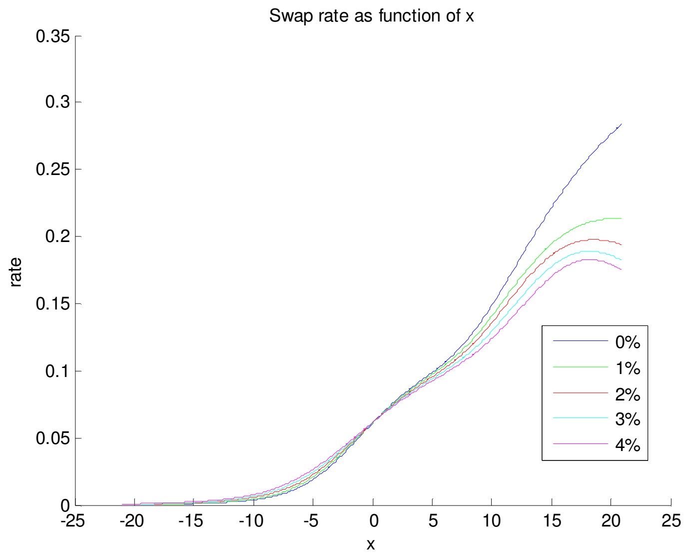
In the next two diagrams we present the “evolution” of the distributions of the swap rates (2x10..11x10), this we can see as an evolution of the 10 years swap rate stochastic process:





We can clearly see how the variation of (marginal distributions of) the process is increasing, and how the shape of the distribution is evolving.

We have also looked at the monotonicity "deterioration" with the increasing mean-reversion with 11x10 rate:



Indeed the calibration for the higher mean reversion is an issue.

8.3.2 Simulation

Using the Monte Carlo method in our SMF model actually means approximating the evolution of (7.1.1), here in its equivalent integral notation:

(8.3.2.1) $$x(t) = \int_0^t e^{as} dW_s$$

by its discrete equivalent, actually an Euler 1st order approximation:

$$x(t + \Delta t) = x(t) + e^{at}(W_{t+\Delta t} - W_t)$$

where the increment of the Brownian Motion are just independent normal variables, thus:

(8.3.3.2) $$x(t_{i+1}) = x(t_i) + e^{at_i}\sqrt{t}\varepsilon_i : \varepsilon_i \sim N(0,1), \forall i$$

The choice of Δt, the "mesh size", is quite critical here. In some first attempts to valuate the calibrating instruments in SMF-MC model we were consistently overpricing them, having more or less constant positive bias. This one disappeared when we reduced the time step with factor 10. In fact we were thus overestimating the integral in (8.3.2.1) by taking too large steps in (8.3.3.2).

We have then simulated M such sample paths of x, and for each run we have calculated the payoff of the instrument we would like to simulate, using the functional form (7.2.1) for a rate in consideration, e.g. for a swaption maturing in 9, where the payoff function is (with k the strike):

$$V = (y_9 - k)^+$$

Then from the central limit theorem we know that the random variable V^* (where $V^{(j)}$ is the payoff from the j-th simulation run):

$$V^* = \frac{1}{M}\sum_{j=1}^{M} V^{(j)}$$

converges for large M to a normal distribution with mean $\mathbb{E}V$ and variance $\frac{1}{M}Var(V)$, thus with increasing M the random variable V^* becomes ever more accurate estimate of $\mathbb{E}V$.

The expectation $\mathbb{E}V$ is actually the expectation under the SMF terminal measure, thus the expectation with the probability measure where we use the terminal discount bond D_N as numeraire and all contingent claims divided by such numeraire are martingales in this measure, hence:

$$\frac{V_0}{D_N(0)} = \mathbb{E}^N\left[\frac{V_n}{D_N(T_n)}\right]$$

where n is the maturity (payoff time) of our to-be-priced instrument, therefore for each simulation run we have to bring the payoff in the year n "up to" the terminal year N, in other words: expressing the payoff in the year-n euro's in the year-N euro's:

$$V^{(j)} \rightarrow \frac{V^{(j)}}{D_N(T_n)} \equiv V_N^{(j)}$$

which is exactly what happens if we use the Markov functional form of (7.2.1), like in the Discount Bond put payoff function (7.4.1), and then we can conclude that for the current price of our instrument:

$$V_0 \approx D_N(0).V^*$$

with $V^* = \frac{1}{M}\sum_{j=1}^{M} V_N^{(j)}$ now.

To indicate the "quality" of the MC (approximation) price we also calculate the standard deviation of V^*, which is called the *standard error* of the Monte Carlo simulation:

$$\text{stderr}(V^*) = \sqrt{\frac{\sum_{j=1}^{M}(V^{(j)})^2 - M.(V^*)^2}{M(M-1)}}$$

The results of the simulations for one calibrating instrument (the rest are similar) are here, where:

- **k**=strike (with 6% is ORA approximately at the money),
- **a**=mean-reversion level,
- **MC 11x10**=swaption on 10 years swap maturing in year 11,
- **stderr**=Monte Carlo standard error,
- **fit**=maximum fitting error for all calibrating instruments,
- **CB 11x10**=instrument equivalent with MC 11x10, actually a put on the coupon bearing bond,
- **Black 11x10**=calibrating Black's price of the swaption.

Table 8.4.1 SMF-MC Calibration check

k	a	MC 11x10	stderr	fit	CB 11x10	stderr	fit	Black 11x10
3%	0%	0,070661	0,000135	0,012758	0,070659	0,000134	0,009929	0,070649
3%	1%	0,070952	0,000134	0,303275	0,070799	0,000132	0,150499	0,070649
3%	2%	0,071030	0,000134	0,381608	0,070747	0,000131	0,098881	0,070649
3%	3%	0,071106	0,000134	0,457839	0,070714	0,000130	0,064962	0,070649
3%	4%	0,071190	0,000134	0,541879	0,070700	0,000129	0,051094	0,070649
4%	0%	0,070670	0,000135	0,021317	0,070668	0,000134	0,019389	0,070649
4%	1%	0,070838	0,000134	0,189722	0,070690	0,000132	0,041869	0,070649
4%	2%	0,070934	0,000134	0,285785	0,070648	0,000131	-0,000718	0,070649
4%	3%	0,071162	0,000134	0,512985	0,070777	0,000130	0,128820	0,070649
4%	4%	0,071296	0,000134	0,647858	0,070805	0,000129	0,155976	0,070649
5%	0%	0,070623	0,000135	-0,025575	0,070623	0,000134	-0,025326	0,070649
5%	1%	0,070782	0,000134	0,133072	0,070639	0,000132	-0,009550	0,070649
5%	2%	0,071028	0,000134	0,379231	0,070741	0,000131	0,092446	0,070649
5%	3%	0,071225	0,000134	0,576543	0,070824	0,000130	0,175464	0,070649
5%	4%	0,071278	0,000134	0,629521	0,070781	0,000129	0,132604	0,070649
6%	0%	0,070686	0,000135	0,037624	0,070697	0,000134	0,048506	0,070649
6%	1%	0,070804	0,000134	0,155371	0,070650	0,000132	0,001505	0,070649
6%	2%	0,071005	0,000134	0,355906	0,070716	0,000131	0,067744	0,070649
6%	3%	0,071178	0,000134	0,529481	0,070782	0,000130	0,133327	0,070649
6%	4%	0,071256	0,000134	0,607526	0,070760	0,000129	0,111166	0,070649
7%	0%	0,070693	0,000135	0,044208	0,070698	0,000134	0,049128	0,070649
7%	1%	0,070825	0,000134	0,176423	0,070680	0,000132	0,031676	0,070649
7%	2%	0,071018	0,000134	0,369305	0,070738	0,000131	0,088941	0,070649
7%	3%	0,071116	0,000134	0,467717	0,070731	0,000130	0,082890	0,070649
7%	4%	0,071248	0,000134	0,599390	0,070763	0,000129	0,114204	0,070649
8%	0%	0,070667	0,000135	0,018033	0,070664	0,000134	0,015635	0,070649
8%	1%	0,070810	0,000134	0,161611	0,070653	0,000132	0,004802	0,070649
8%	2%	0,070809	0,000134	0,160620	0,070541	0,000131	-0,107547	0,070649
8%	3%	0,071088	0,000133	0,439633	0,070703	0,000130	0,054465	0,070649
8%	4%	0,071269	0,000134	0,620266	0,070783	0,000129	0,134305	0,070649

We see that the fit is really satisfactory, nevertheless the fit error is almost always positive, meaning that we still have slight positive bias (overpricing). This bias did not disappear with yet smaller time-step size in the sample paths simulations, this remains a small issue.

Next comes "the cherry on the pie", our ORA option:

- **SMF**=SMF-MC price of the ORA option,
- **Levy**=Levy's approximation using the same simulated covariance's and variances, effectively adapting to volatility smile,
- **%Diff1**=percent SMF ORA price from Levy price
- **SMF comp**=SMF-MC ORA price now with compensation as described in 4.3,
- **Levy flat/comp**=Levy's approximation exactly as described in 4.4, thus with rudimentary Black's correlation structure and flat volatility, this single flat volatility was calibrated to give the same price as "ORA comp" with k=6% and a=0%.
- **%Diff2=**percent SMF compensated ORA price from Levy price
- **%Diff3**=percent SMF ORA non compensated versus compensated

Table 8.4.2 ORA pricing

k	a	SMF	stderr	Levy	%Diff	SMF comp	Levy flat/comp	%Diff2	%Diff3
3%	0%	0,171600	0,000206	0,168059	102%	0,171597	0,165572	104%	100,00%
3%	1%	0,172423	0,000202	0,168135	103%	0,172420	0,165572	104%	100,00%
3%	2%	0,172841	0,000199	0,168085	103%	0,172838	0,165572	104%	100,00%
3%	3%	0,173301	0,000196	0,168120	103%	0,173298	0,165572	105%	100,00%
3%	4%	0,173643	0,000193	0,168091	103%	0,173640	0,165572	105%	100,00%
4%	0%	0,116135	0,000177	0,115476	101%	0,116121	0,108895	107%	100,01%
4%	1%	0,116394	0,000173	0,114923	101%	0,116380	0,108895	107%	100,01%
4%	2%	0,116597	0,000170	0,114320	102%	0,116582	0,108895	107%	100,01%
4%	3%	0,116893	0,000168	0,113894	103%	0,116878	0,108895	107%	100,01%
4%	4%	0,116979	0,000166	0,113504	103%	0,116963	0,108895	107%	100,01%
5%	0%	0,069407	0,000141	0,074955	93%	0,069372	0,064431	108%	100,05%
5%	1%	0,069375	0,000137	0,073824	94%	0,069339	0,064431	108%	100,05%
5%	2%	0,069408	0,000136	0,072682	95%	0,069371	0,064431	108%	100,05%
5%	3%	0,069323	0,000134	0,071847	96%	0,069284	0,064431	108%	100,06%
5%	4%	0,069120	0,000132	0,071067	97%	0,069080	0,064431	107%	100,06%
6%	0%	0,035001	0,000102	0,046739	75%	0,034943	0,034940	100%	100,17%
6%	1%	0,034760	0,000098	0,045374	77%	0,034701	0,034940	99%	100,17%
6%	2%	0,034630	0,000097	0,043987	79%	0,034570	0,034940	99%	100,17%
6%	3%	0,034434	0,000095	0,042942	80%	0,034372	0,034940	98%	100,18%
6%	4%	0,034187	0,000094	0,042024	81%	0,034122	0,034940	98%	100,19%
7%	0%	0,014043	0,000066	0,028436	49%	0,013977	0,017790	79%	100,47%
7%	1%	0,013779	0,000062	0,027126	51%	0,013713	0,017790	77%	100,48%
7%	2%	0,013634	0,000061	0,025788	53%	0,013568	0,017790	76%	100,49%
7%	3%	0,013406	0,000059	0,024762	54%	0,013338	0,017790	75%	100,51%
7%	4%	0,013210	0,000058	0,023951	55%	0,013141	0,017790	74%	100,53%
8%	0%	0,004348	0,000038	0,017048	26%	0,004298	0,008686	49%	101,16%
8%	1%	0,004070	0,000034	0,015901	26%	0,004018	0,008686	46%	101,29%
8%	2%	0,003892	0,000032	0,014821	26%	0,003839	0,008686	44%	101,38%
8%	3%	0,003782	0,000031	0,013959	27%	0,003729	0,008686	43%	101,42%
8%	4%	0,003721	0,000030	0,013317	28%	0,003669	0,008686	42%	101,42%

Notice that setting all tranches to 1 in the option's payoff (8.1.1) is equivalent with setting the option's nominal to 1, meaning for example that with the strike of 4% the SMF and Levy's ORA prices are approximately 11% of the nominal.
In the following sections we analyze some effects seen in this table.

8.3.3 Mean-reversion and ORA price

With increasing mean-reversion of our driving process x:

$$Corr(x(t), x(s)) = \sqrt{\frac{e^{2at} - 1}{e^{2as} - 1}} \quad ,t < s$$

we see that the correlation between the swap rates should decrease with increasing mean-reversion because, although the rates are non-linear, they are monotone functions of x, and then from the following equality:

$$Cov(y_t, y_s) = \sigma_t \sigma_s Corr(y_t, y_s)$$

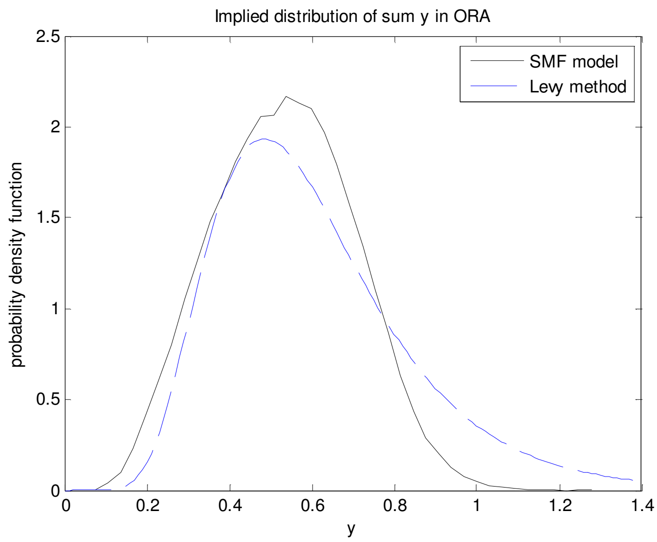
it's clear that the covariance decreases too, as the SMF calibration procedure should take care that the standard deviations of the marginal distributions of y remain the same, the fact that we have already validated in section 8.3.1.

Now as the covariance decreases, the variance (hence also volatility) of the sum of the rates in the ORA payoff will fall too, as:

$$Var(y_t + y_s) = Var(y_t) + Var(y_s) + Cov(y_t, y_s)$$

and with it the price of ORA should fall too, at least in the Black's world.

This seems to be the case for the Levy's approximation ("Levy" column in the table 8.4.2), although not yet with strike of 3%. The SMF price ("SMF" column) starts showing this effect from k=5% further on. Let's see about this supposedly decreasing variance of the sum of rates in our MC simulation and draw the distribution of the sum of the rates involved in ORA payoff, for mean-reversion 0%:

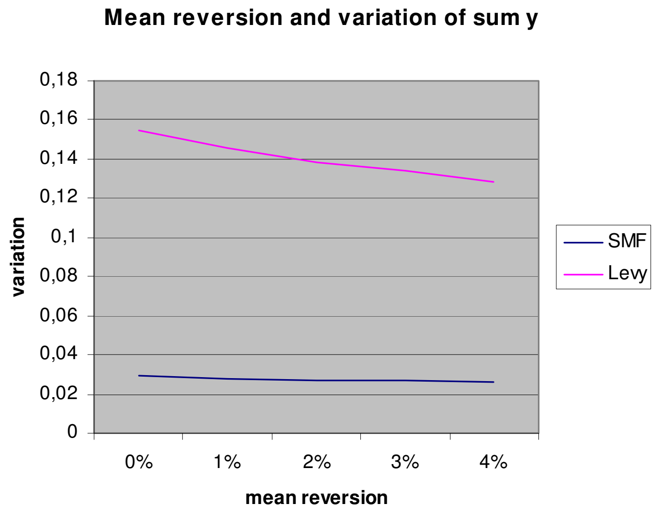


The variations are 0.0291 for SMF and 0.1549 for Levy, which is also clear from the diagram, as the Levy's implied distribution is just lognormal and thus "wider".

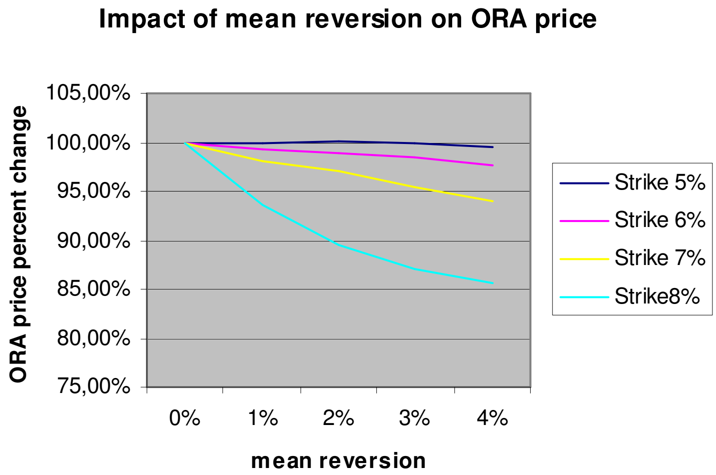
Let's increase the mean-reversion now and see about $Var(\sum y)$:

Mean reversion	SMF	Levy
0%	0.0291	0.1549
1%	0.0280	0.1457
2%	0.0272	0.1381
3%	0.0267	0.1341
4%	0.0261	0.1288

Stated graphically:

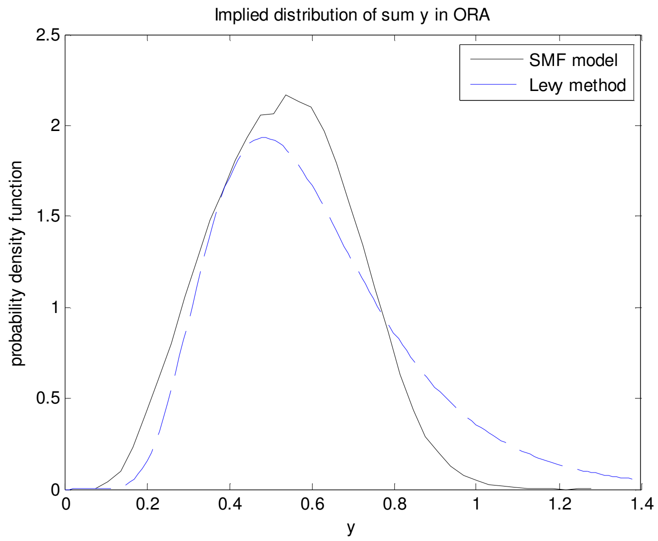


It's clear that the effect of increasing mean reversion is much bigger by Levy compared to SMF and that's why the price of ORA by Levy is going down so nicely, while ORA does this only with strikes above 6%, but then it's quite consistent:



8.3.4 ORA price SMF against Levy (without compensation)

In the table 8.4.2 we also see that the SMF price is generally higher than Levy, but for strike of 3% they are 97% the same ("%Diff1"column in the table 8.4.2). For higher strikes, like 7%, the accuracy is only about 50%, getting down to 30% with strike 8%. Looking at the implied distribution of the sum of rates again:



we see that the difference is expected with higher strikes as the Levy has much thicker tail there.

8.3.5 ORA price with and without compensation

For the price with the compensation mechanism in place, and where Levy is applied using only flat volatility and Black's correlation, the prices are surprisingly closer together, as opposed to the case without the compensation. The differences lie in the range from 108% down to 42% as opposed to 103% to 26%.

The effect of the compensation itself looks surprisingly small for the SMF ORA price; see the "%Diff3" column in the table. Statistically seen the compensation mechanism decreases the price of the ORA option only with about 1% in our simulations.

9 Conclusion

Besides known wisdoms like "In the long term we're all dead" we can still draw a couple of conclusions about the valuation of the long-term/long-maturity products like ORA in the framework of the SMF-MC method.

The method seems viable for valuation of such products even in the real market environment, as we have used non-flat initial term structure and volatility smile in our model. The calibration is nevertheless still an issue, but the original SMF method itself provides a remedy, which is yet to be tested of course.

The relation between the mean-reversion level and the ORA option price is also worth of further research, the result are quite inconclusive, it's not clear if we can blame this on the bad calibration, the consistent positive bias on the SMF prices of the calibrating instruments seems to support this. In fact the question is if the falling option's price with the increasing mean reversion (and thus falling volatility) is something specific to the Black and Scholes (lognormal) world.

Another interesting point is that the SMF and Levy prices are quite close in the range of the "normal" strikes, it is not to be expected that the bank selling the ORA product would set up the guaranteed rate completely out-of-the-money or in-the-money. We might thus conclude that Levy does quite a good job in his approximation.

Surprisingly enough the compensation mechanism seems to have a very little effect on the ORA price in our SMF-MC framework as opposed to Levy.

And last but not least the possible future research areas could be summarized as follows:

- review the relation between the mean reversion and the ORA price in SMF model,
- try to improve the calibration process through the usage of the original SMF method, thus with the Jamshidian decomposition of a swaption, followed by the analytical integration, plus guaranteed monotonicity and positivity at each iteration,
- we need to test the model on the real market data, although our test-environment was as close as possible to it,
- we did not tackled the hedging problem at the end, but in our MC framework is should be straightforward,
- it would be interesting to valuate some complex but still liquid instrument like CMS floors in this framework.

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