Collateralized Debt Obligation (CDO) modelling and its model comparisons

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BMI-paper

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Abstract

Since CDO pricing issue attracts more attention these years, many researchers are devoted themselves to the pricing model studies. In practice, standard Gaussian copula model becomes the market standard in the financial filed. However, due to its weakness of mispricing, many extension models are researches, such as student t copula, Clayton copula, factor loading Gaussian copula, implied copula approach etc. They all show improvements to the Gaussian copula in terms of fitting to the market quotes, yet with different performances. This paper mainly addresses the pricing models for CDO tranche available so far, and presents its model comparisons with merits and disadvantages.

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1. Introduction

With the market of credit derivatives grows larger, collateralized debt obligations (CDOs) as one of their most popular instruments also gain more interests both from the market side and the academic side. Though it appears a dramatic increase in the traded CDO contracts, unfortunately, achieving a precise CDO tranche valuation is still a difficult and an open issue today.

This paper explores the vast area of the credit derivatives and the literatures on the pricing models of collateralized debt obligations (CDOs). I aim at gaining insight in the financial as well as mathematical foundation of the credit derivatives and CDO pricing models. In order to obtain the fair premium of the CDO tranche, the market standard pricing model---one factor Gaussian copula model, and its various extension models (e.g. student t, double t, factor loading, normal inverse copula, perfect copula etc) are presented. Besides considering the model selection, I also discuss about the comparisons of model performances, its advantages and weakness.

The rest of the paper is structured as follows: chapter 2 gives to a general overview of credit derivatives. In particular, some of the most popular credit derivative instruments are addressed. Chapter 3 focuses on the basic knowledge about the synthetic CDO tranche and its valuation methodology using general semi-analytic approach. Particularly, I discuss a few concepts and their relationships, such as loss distribution, large portfolio approximation and default correlation. They are crucial for calculating the fair premium of CDO tranche. In chapter 4, I present the copula function and default correlation models (factor copula model), which constructs a general framework for various pricing models discussed in the next few chapters. Then, chapter 5 elaborates on the market standard pricing model---one factor Gaussian copula model, including its conditional/unconditional loss distribution functions, large portfolio approximations and evaluation issues. It is argued that despite Gaussian copula model is widespread deployed in the market nowadays; they do show many shortcomings. Some of those even lead to serious consequence of huge mispricing discrepancy. Subsequently, the possible reasons for mispricing are also presented, such as the well-known correlation smile. Considering the insufficiencies of the market standard model, various extension copula models are presented in chapter 6, which aim at reproducing the correlation skew and fit the market quote better. In chapter 7, model comparisons based on abilities of the different models to reproduce market quotes are discussed. Finally, conclusions are drawn in chapter 8. It shows that most extension models have improvements compared to Gaussian copula models, particularly for the following three copula models display best fit the market quote. They are 'factor loading Gaussian copula model', 'normal inverse Gaussian copula model', and the implied copula approach (or perfect copula).

2. Credit derivatives

2.1 Brief overview

Credit derivatives were introduced to the market at the beginning of the 1990's. Despite their short history, their uses have grown rapidly. They are now used not only by banks, but also by various funds, insurance companies, and even corporations.

By definition, 'credit derivatives are a group of financial instruments that have as their common main purpose the managing of credit exposures, and thus credit or default risk' (Jonathan Batten, Warren Hogan, 2002).

As a very useful tool, credit derivative enable the investors to transfer and diversify credit risk. Specifically, for the lenders, such as a commercial bank, who want to reduce their exposure to a particular borrower, but are unwilling to sell their ownerships of underlying assets to the borrower, credit derivatives contracts may be a wise choice. Since it successfully realizes the function of transferring the credit risk without actually transferring the ownership.

Credit derivatives come in many shapes and sizes, and there are several ways of grouping them. Here I introduce the primary category: single-name versus multi-name credit derivatives. Single-name credit derivatives are those involving protection against the default by a single reference entity, such as a credit default swap (CDS). Multi-name credit derivatives are the contracts that are contingent on default events by a pool of reference entities. A simple example is the portfolio default swaps, and the collateralized debt obligations (CDOs).

2.2 Instruments

In the vast area of the credit derivatives world, there are many types of products. This chapter is devoted to the overview of several important credit derivative instruments based on the background knowledge in Bomfim, A.N. 2005.

2.2.1 Credit default swap

Credit default swaps are the most common type of credit derivatives. It is a contract between a protection buyer and a protection seller, whereby the buyer pays a periodic fee (CDS premium) to the seller. In return, the seller will pay a contingent payment to the buyer once credit events happen in the reference entity.

To elaborate, in Figure 2.1, the CDS contract consists of 2 parties; one is the credit protection buyer, the other is the protection seller. In its simplest form, the protection buyer agrees to make periodic payments over a predetermined number of years (referred to as the maturity of the CDS) to the protection seller. In exchange, the protection seller commits to making a payment to the buyer in the event of a default by a third party (referred to as the reference entity). This payment needs a settlement choice specification upfront when entering the contract. In case of a default, payments can be settled physically or in cash. In the physically-settled situation, the protection buyer has the right to deliver a range of defaulted physical assets to the protection seller, receiving the full face value of the assets as payment. And the types of the deliverable assets should be pre-specified in the contract. In the cash-settled situation, payments should be paid in cash and are proportional to the notional amount. Nowadays, the cash settlement is more commonly used in Europe than in the United States, where by far, the majority is physically delivered (Bomfim, A.N. 2005).

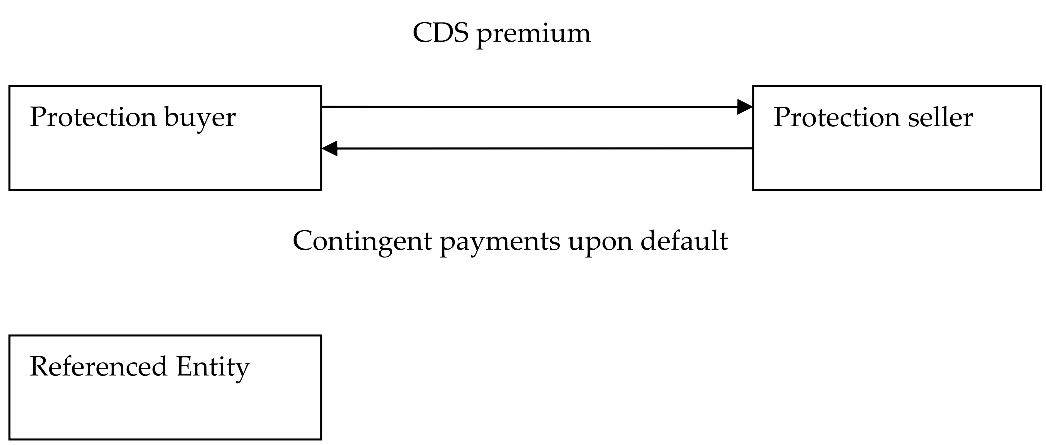


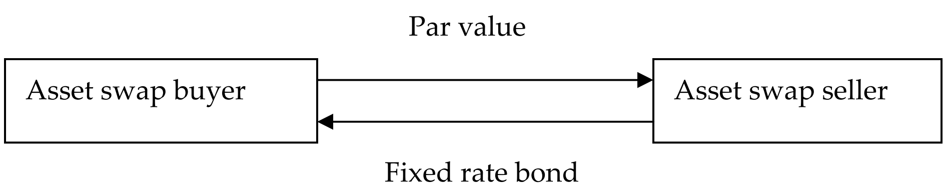
Figure 2.1 CDS structure

2.2.2 Asset swaps

The asset swap is a common form of a derivative contract[1], in which, an investor (asset swap buyer) can buy a fixed rate liability (usually a coupon bond) issued by a reference entity and simultaneously enter an interest rate swap, where the fixed rate and maturity date exactly match those of the fixed-rate liability. At the maturity date, the investor of the asset swap effectively transfers the interest rate (market) risk of the fixed rate liability to its asset swap counterparty (or dealer or asset swap seller), retaining only the credit risk component. As such, we obtain one important characteristic of asset swaps: allowing investors to take pure credit positions. In other words, asset swaps can be used for investors who are willing to take exposure to credit risk without worrying about the interest risks.

To elaborate, we can decompose the process into the following two parts:

- Purchasing the fixed rate bond at par value



The investor (asset swap buyer) agrees to buy from the dealer (asset swap seller) a fixed rate bond issued by the reference entity, paying for the par value regardless of the market value.

- Entering the interest rate swap

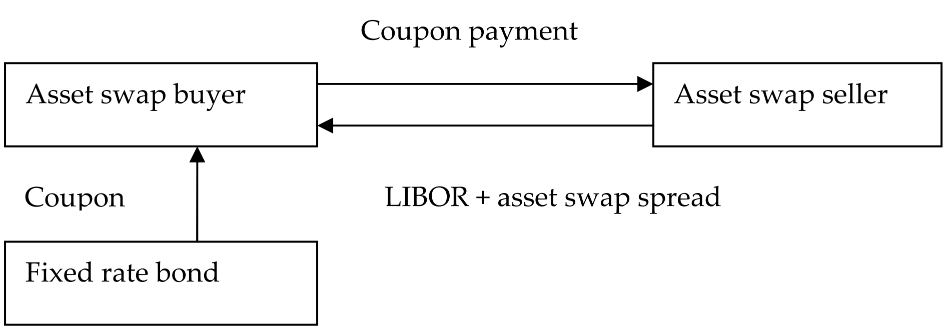


Figure 2.2 asset swap

The investor (asset swap buyer) pays for the dealer's (asset swap seller) periodic fixed rate payments, which are equal to the amount of the coupon paid by

[1] so far, there is still some disagreement on whether the asset swap is a credit derivative

reference bond (fixed rate bond). In return, the asset swap seller will make variable interest rate payments to the investor, which is equal to the amount: Libor plus asset swap spread[2]. In such a way, investor successfully transfers the interest risk retaining merely the credit risk.

2.2.3 Total return swaps

A total return swap (TRS) is a contract that allows investors (the total return receiver/buyer) to obtain all of the economic benefits of owning an asset without actually holding that physical asset. It transfers the returns and risks on an underlying reference asset from one party to another. A total return swap involves a "total return buyer," who pays a periodic fee to a "total return seller"; meanwhile, the total return buyer will receive all the economic benefits of the underlying reference asset in return. The term "total return" actually includes all interest payments on the reference asset plus an amount based on the changes in the asset's market value. At trade inception, one party, the total return buyer, agrees to make the periodic payments of LIBOR plus a fixed spread to the other party, the total return receiver/payer, and in return the buyers will get coupons from some specified asset. At the end of the total return swap, the total return buyer pays the difference between the final market price of the asset and the initial price of the asset. Specifically, if the price goes up, the total-return buyer gets an amount (pay a negative value) equal to the appreciation of the value, and if the price declines, the buyer pays an amount equal to the depreciation in value. If a credit event (like a default) occurs prior to maturity, the TRS usually terminates, and a settlement is made immediately. (Lehman Brothers International (Europe), March 2001)

Unlike the asset swaps, which essentially focus on the credit risk, a total return swap exposes investors to all risks associated with the reference asset, like credit risk, interest risk etc. In addition, an asset swap involves the actual purchase of the asset, which is another difference between the asset swap and the total return swap. Therefore, people may choose to make use of different credit derivatives to diversify risks according to their actual needs.

In the following, the diagrams show how the total return swap works.
During swap:

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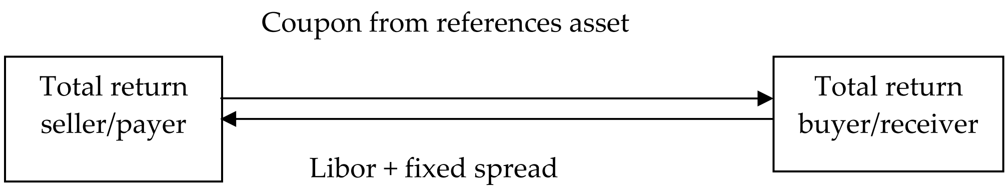
[2] Asset swap spread: The floating rate in such an interest rate swap is conventionally quoted as a spread over short-term LIBOR.



At maturity:

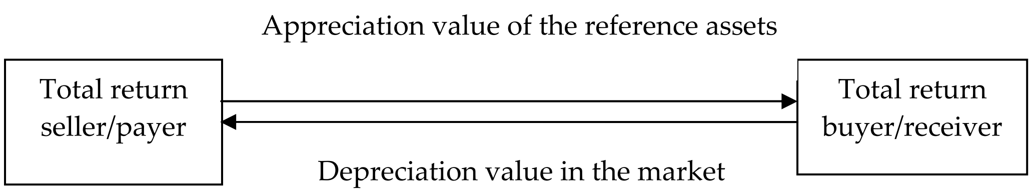


Figure 2.3 Total return swap

2.2.4 Credit linked notes

A credit linked note is a security issued by a special purpose company (bank, etc.), designed to offer CLN buyers (investors/asset managers) periodic coupon and principle payments unless defaults happen. Loosely, the CLN usually links with CDS contracts, which realize the passing of the credit risk on a specified reference entity (in the CDS part) onto CLN investors who are willing to bear that risk in return for the higher yield it makes available.

For specific purposes, the investor who is willing to bear credit risks based on the reference entities pays for the par value to the CLN issuers (special purpose company or dealer) to buy the CLN. For the dealers who issue the credit linked notes, meanwhile, enter into a CDS contract and sell protection against default by the reference entity to the protection buyer. As such, in the process, the investor pays for the par value and the CLN issuer/dealer pays a fixed- or floating-rate coupon in return. Notice that the investors retain an exposure to the reference entity, which means that in case a default happens, investors have to bear the full brunt of the lose, including some or all of their coupon and principal, yet receiving a sum of money based on the recovery rate. (Lehman Brothers International (Europe), March 2001). Specifically, in the event a default takes place, the dealer pays its CDS counterparty the value equal to the notional amount multiplied by (1- recovery rate), and meanwhile the CLN is terminated with the investors receiving only (notional amount * recovery rate).

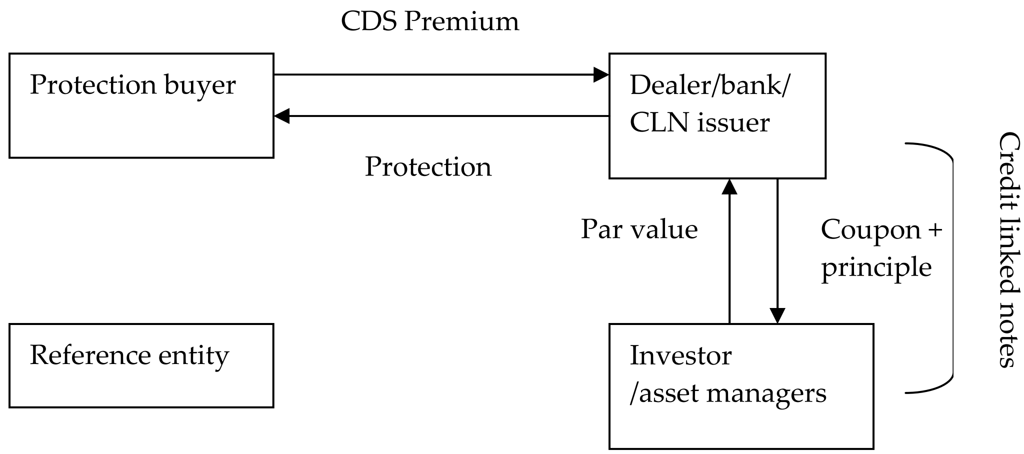


Figure 2.4 Credit linked notes

2.2.5 Collateralized Debt Obligation

A collateralized debt obligation (CDO) is defined as a structured financial product backed by portfolios of assets. Those assets are called collateral, which usually includes a combination of debt instruments or loans, such as bonds, loans, asset-backed securities, etc. When the collaterals are loans, the CDO is called a collateralized loan obligation (CLO); if they are bonds, it turns to be a CBO (collateralized bond obligation). A CDO has a sponsoring organization, which sets up a specially created company, namely a *special purpose vehicle* (SPV) in order to hold the collateral and issue securities to investors. The sponsoring organization may contain sponsoring banks and other financial institutions. There are multiple tranches of securities issued by the CDO (SPV), offering investors various maturity and credit risk characteristics based on the collaterals assets. And according to the level of credit risk, tranches are then categorized as senior, mezzanine, and subordinated/equity (from the lowest to the highest degree of credit risk). If there are defaults or underperforms of the CDO's collateral, the investor who buys the equity tranche has to first bear the loss, and then the mezzanine and the senior tranche. In other words, the payments to senior tranches take precedence over those of mezzanine tranches, and the payments to mezzanine tranches take precedence over those to subordinated/equity tranches. Essentially, by selling these securitized collateral assets in the form of tranched securities, the issuer (SPV) transfers the complete credit risk of the collateral pool to the investors. To elaborate, a simple CDO example with a figure is made in the following.

Consider that a CDO issuer buys a portfolio of bonds as collateral, [3]which has a total face value of, say 100 million euros. To fund the purchase of bond portfolios, the issuers sell the debt obligation notes (tranche securities) to investors. In our case, the CDO issuer is the SPV.

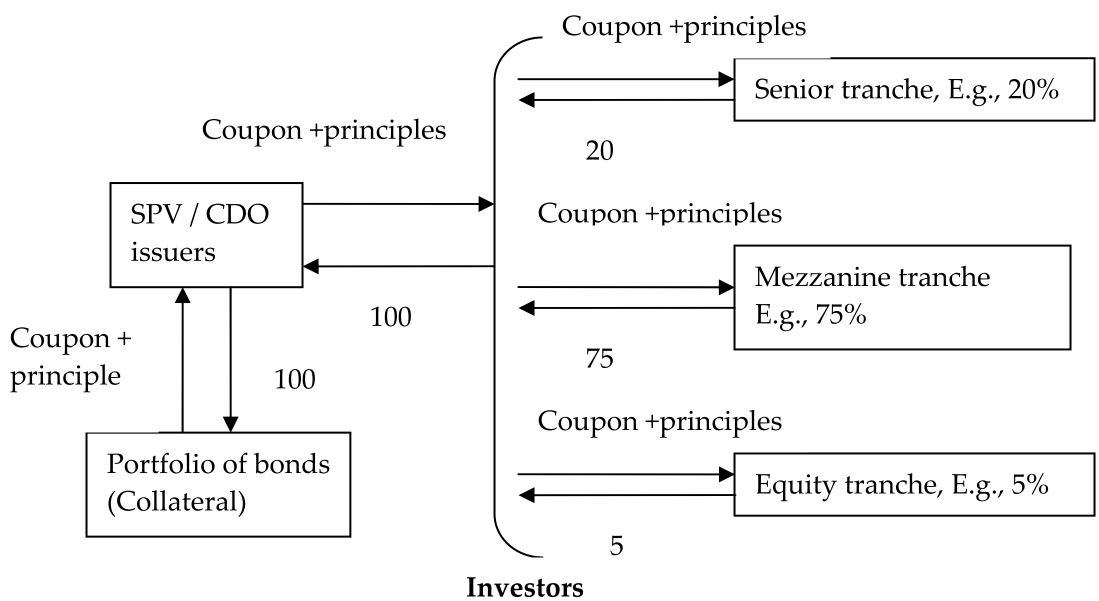


Figure 2.5 structures of simple cash CDO
Note that in Figure 2.5, all numbers are in million euros.

Given that the collateral comprise of the portfolio of bonds and the debt obligation notes make monthly payments. Each month, the SPV receive the payments (coupons) from the bonds and pass them through to the investors who purchase the notes. In particular, the payments from the bonds have to first meet the amount owed to the most senior notes/ tranche holder. And then the second most senior notes/ tranche holder is paid up, and so on until the most junior notes/ tranche holder receive their share of cash flow. In the event of a default by the bonds in the portfolio, however, the most junior notes holder may receive less than their total payments in the case of no defaults. Because the amount they receive in the default situation is essentially the residual amount after the more senior and junior investors are paid.

There are several ways to classify a CDO. Here I introduce the most widespread category: the synthetic versus cash CDOs. So far, I have been discussing about the cash CDOs. Cash CDOs expose investors to the credit risk by actually

[3] As discussed above, we may call it a CBO. But this example would work just as well as a CLO.

holding the collateral that is subject to defaults. By comparison, the synthetic CDO holds the high quality or cash collateral that has little or no default risk. It exposes investors to credit risk by adding credit default swaps (CDSs) to the collateral. So the synthetic CDO is actually backed by serials of single name CDS. A simple example with diagram is made as following.

Consider a commercial bank (as sponsoring bank in the diagram) with a bond portfolio of 100 million euros, which can be seen as the reference assets.

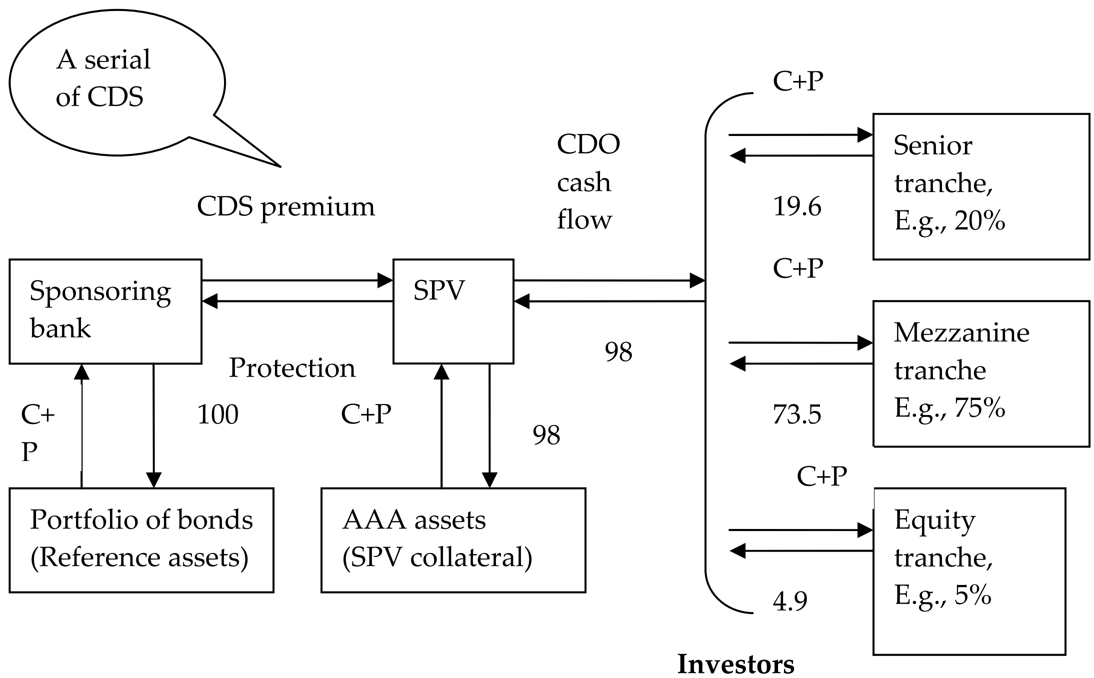


Figure 2.6 Synthetic CDO structure

Note that in Figure 2.6, all numbers are in million euros.

C+P : means coupon + principles

The bank wants to transfer its credit risks associated with the bond portfolio, but is not willing to sell them to the SPV. In other words, the bank tends to sell only the credit risk related to the bond portfolio (reference assets) and still keep the bonds in its balance sheet. The transfer of the risk is carried out by a serials of CDS, where the SPV is the counterparty and where the sponsoring bank buys a protection of any loss in excess of, say, 2% of the portfolio. As in the cash CDO structure, the SPV then issues the notes to various classes of investors (3 in the example). In the synthetic CDO structure, however, because the premiums payments by the sponsoring bank cannot fully compensate all the costs that the SPV paid to the investors, another funding source is needed. To make up for the shortfall, the SPV invests its proceeds of the notes issuing in high grade assets, typically AAA-rated instruments, which then are employed as both the collateral for the obligations towards the sponsoring banks and the supplements of the coupon payments promised by the notes.

If no defaults happen, at the maturity date of CDO notes, the CDS is terminated and the SPV then liquidates the collateral to repay the investor's principles in full. In the event of defaults, the CDO investors have to absorb all the defaulted related loss in excess of the part retained by the sponsoring bank, in our example 2%. More details about the synthetic CDO tranches will be discussed in the next chapter.

3. Synthetic CDO tranche and its valuation

In the previous chapter, I have discussed about several types of credit derivatives, especially about the CDOs, the structure and principles of the cash versus synthetic CDOs. In the rest of my paper, I would like to primarily focus on synthetic CDOs, its basic knowledge, valuation methodology, pricing models and model comparisons, etc.

3.1 Basic knowledge about CDO tranches

As we already discussed, a collateralized debt obligation is a financial instrument that transfers the credit risk of a reference portfolio of assets. Specifically, issuers of a CDO (the SPV) on one side, enter into a serial of single name CDS, provide protection to the sponsoring entity (like commercial banks, etc.) on the default risk of its reference entities; on the other side, SPV issue tranched securities (CDO notes) to investors; in such a way the CDO then passes the default risk of the protection buyers on to the synthetic CDO's investors or call them tranche holders.

The risk of loss or the defaults on the reference portfolio of assets is tranched into different levels. In Table 2.1, an example of various levels of tranches expressed in the form of percentage is displayed. The starting point and the ending point of each tranche level are called the attachment point and the detachment point, respectively. Note that on each level, the detachment point overlaps with the next level's attachment point. For a given tranche level investors (protection sellers) purchase, they have to pay a payoff consisting of all losses/defaults that are greater than a certain percentage (the corresponding attachment point), and less than another certain percentage (the corresponding detachment point) of the notional amount of the reference portfolio assets. In return for the protection, the issuance buyers also pay premiums, typically quarterly, proportional to the *remaining notional amount of reference entities* at the time of payment. The investors who sell their insurance on the tranches could obtain that premium, which is distributed to the tranches in a way that reflects the credit risk they are bearing and that should be specified upfront. For example, the equity tranche which is the riskiest might get 3,000 basis points per annum; the junior tranche might get 1,000 basis points per annum since it is less risky, the third one is even less and so on.

The following table shows standard tranche levels of a synthetic CDO and its attachment/detachment points.

Reference Portfolio	Tranche level	Tranche name	A	D
DJ iTraxx Europe: Portfolio of 125 CDS	1	Equity	0%	3%
	2	Junior Mezzanine	3%	6%
	3	Senior Mezzanine	6%	9%
	4	Senior	9%	12%
	5	Super Senior	12%	22%

Table 3.1: standard structure of a synthetic CDO on DJ iTraxx Europe
A: represents attachment points
D: represents detachment points
Super-Senior: we use 'super' because its credit quality has to be higher than Aaa at inception
Given that the successive tranches are responsible for 0% to 3%, 3% to 6%, 6% to 9%, 9% to 12%, and 12% to 22% of the losses/defaults (which is the case of the synthetic CDO on DJ iTraxx Europe). If default of reference entities takes place, loss occurs. The first tranche/Equity has to absorb all the losses until they reach 3% of the total notional principal; and if the total loss exceeds 3%, the second tranche/ Junior Mezzanine has to bear the rest of losses until they reach 6%; the third tranche/ Senior Mezzanine would then be responsible for the payoff between 6% to 9% of the total notional amount; and so on.

DJ iTraxx instruction
The DJ iTraxx Investment Grade index (DJ iTraxx index) is the main index in the family of CDS index products, which, in Europe, consists of a portfolio of the top 125 names (125 investment grade European companies) in terms of CDS volume traded in the six months prior to the roll. Each name is equally weighted in the static portfolio. And a new series of DJ iTraxx Europe is issued every 6 months. In the case of the iTraxx EUR 5 yr index, successive tranches are responsible for

0% to 3%, 3% to 6%, 6% to 9%, 9% to 12%, and 12% to 22% of the losses. We should note that an index tranche is different from the tranche of a synthetic CDO in that an index tranche is not funded by the sale of a portfolio of credit default swaps. However, the method of pricing the tranche of a CDO ensures that an index tranche is economically equivalent to the corresponding synthetic CDO tranche. More information about DJ iTraxx may be found at www.iboxx.com.

3.2 Valuation methodology of synthetic CDO tranche

After obtaining some ideas on CDO tranches, I would at this section introduce how to price the CDO tranches. More precisely, how much should the SPV pay as premiums (coupons) to the tranche investors[4] (protection sellers). Consider a synthetic CDO; as long as no defaults take place, the SPV pays a regular premium to the tranche holder. In the event of defaults, the investor has to bear the loss. The next premium is then paid on the remaining notional amount, which is original notional amount reduced by the loss amount.

3.2.1 Loss distribution introduction

Recall that the payment under default should absorb between the tranche level's attachment point A and the detachment point D.

Let $N(t)$: the cumulative loss on a given A-D tranche at time t; $t = 1, 2 \cdots m$

$L(t)$: the cumulative loss on the whole reference entities at time t

We get

$$N(t) = \begin{cases} 0 & \text{if } \mathrm{L(t)} \leq A \\ L(t) - A & \text{if } \mathrm{A} < \mathrm{L(t)} \leq D \\ D - A & \text{if } \mathrm{L(t)} > \mathrm{D} \end{cases}$$

Calculation of the portfolio loss $L(t)$ and portfolio loss on a given tranche level $N(t)$ are important, since they are essential elements to decide the amount of contingent payments (expected cumulative losses) as well as the cash flows between the protection buyers and sellers, hence to obtain the premium of a CDO tranche.

Consider N references entities/ companies /obligors with notional amount A_i, recovery rate R_i. Let $B_i = (1 - R_i)A_i$, $i = 1, 2, \cdots N$, be the losses given the obligor i;

[4] Bear in mind that the CDO buyers or CDO investors are actually the tranche holders; they are standing the role of the protection sellers or the risk takers. The protection buyers are usually the sponsoring banks ,yet they are not directly paying the premiums to investors. They pay coupons to SPV, and it in principle is SPV who pays the tranche premiums to investors.

Let η_i be the default time of company i;
Let

$$H_i(t) = \begin{cases} 1 \text{ if } \eta_{\text{i}} < t \\ 0 \text{ otherwise} \end{cases} \text{ be a counting process,}$$

Define: $H_i(t)$ turns to be 1 when the default time of company i is smaller than time t.
Thus the whole portfolio loss at time t is:

$$L(t) = \sum_{i=1}^{N} B_i H_i(t)$$

We assume that A_i and R_i are the same for all obligors, then B_i is constant. Given the time t discrete loss of the reference portfolio $L(t)$ with probability p_t , $t = 1, 2 \cdots m$. The A-D CDO tranche suffers a loss of $\{\min(L(t), \text{ D}) \text{ - A}\}^+$ with the probability p_t , $t = 1, 2 \cdots m$. Then the expected cumulative loss or contingent payment on a given A-D tranche in the case of **discrete loss distribution** is

$$EL_{(A,D)} = \frac{E[N(t)]}{D-A} = \frac{1}{D-A} \sum_{t=1}^{m} \{\min(L(t), \text{ D}) \text{ - A}\}^+ p_t \qquad (3.1)$$

In the case of **continuous portfolio loss distribution function** $F(x)$**,** the expected cumulative loss or contingent payment on a given tranche is

$$EL_{(A,D)} = \frac{1}{D-A} \{ \int_A^1 (x-A) dF(x) - \int_D^1 (x-D) dF(x) \} \qquad (3.2)$$

Proof:

$$EL_{(A,D)} = \frac{1}{D-A} \sum_{t=1}^{m} \{\min(L(t), \text{ D}) \text{ - A}\}^+ p_t$$

$$= \frac{1}{D-A} \sum_{t=1}^{m} [L(t) \cdot 1_{\{L(t)<D\}} + D \cdot 1_{\{L(t) \geq D\}} - A] \cdot 1_{\{\min(L(t),D)>A\}} \cdot p_t$$

$$= \frac{1}{D-A} \sum_{t=1}^{m} [(L(t) - A) \cdot 1_{\{A<L(t)<D\}} + (D-A) \cdot 1_{\{L(t) \geq D > A\}}] \cdot p_t$$

$$= \frac{1}{D-A} \{ \int_A^D (X-A) dF(x) + \int_D^1 (D-A) dF(x) \}$$

$$= \frac{1}{D-A} \{ \int_A^1 (X-A) dF(x) - \int_D^1 (X-A) dF(x) + \int_D^1 (D-A) dF(x) \}$$

$$= \frac{1}{D-A} \{ \int_A^1 (x-A) dF(x) - \int_D^1 (x-D) dF(x) \}$$

Note that all expectations are calculated under the risk neutral measure Q and the expectation values are smaller than 1 in the form of percentage (Davide Meneguzzo,Walter Vecchiato (2004)).

3.2.2 Pricing synthetic CDO tranche using General semi-analytic approach

In the previous section, a few important variables and elements, such as the contingent payments for a given tranche level A-D[5] are already derived; this section will focus on the popular valuation methodology --- General semi-analytic approach. Its basic idea is within each trache to construct a premium value so that the expected premium leg equals the expected protection leg. Equivalently, if protection buyer pay for the fair premium to investors, the present value of the spreads payment ('premium leg') should be equal to the present value of the contingent payment $EL_{(A,D)}$ ('protection leg') paid by the protection sellers. $EL_{(A,D)}$ is the variable we already obtained in the section 3.2.1.

Namely:

$$V_{PREM} = V_{PROT}$$

Let's assume that

$$0 \le t_0 < t_1 \cdots < t_{m-1}$$

denote the premium payment dates.

The CDO contracts specify quarterly payments (four times a year) until the maturity date t_m. The value of the premium leg V_{PREM} of a given tranche is the present value of all expected spread payments:

$$V_{PREM} = \sum_{i=1}^{m} \Delta t_i D(t_0, t_{i-1})[1 - EL_{(A,D)}(t_{i-1})] \cdot S \qquad (3.3)$$

Where Δt_i : $\Delta t_i = t_i - t_{i-1}$

$D(0, t_i)$: the discount factor for the time value until time t_i

S : the fair premium/spread (annual basis), in basic point.

$EL_{(A,D)}(t_i)$:the expected loss of A-D CDO tranche at time in term of percentage

Accordingly, the protection leg is calculated as follows:

$$V_{PROT} = \int_{t_0}^{t_m} D(t_0, s) dEL_{(A,D)}(s) \qquad (3.4)$$

$$\approx \sum_{i=1}^{m} D(t_0, t_i)[EL_{(A,D)}(t_i) - EL_{(A,D)}(t_{i-1})]$$

The fair premium S^* can be obtained by solving equation $V_{PREM} = V_{PROT}$

[5] In this chapter, the valuation methodology is discussed for a given tranche level A-D.

Thus

$$\sum_{i=1}^{m} \Delta t_i D(t_0, t_{i-1})[1 - EL_{(A,D)}(t_{i-1})] \cdot S = \sum_{i=1}^{m} D(t_0, t_i)[EL_{(A,D)}(t_i) - EL_{(A,D)}(t_{i-1})] \quad (3.5)$$

$$S^* = \frac{\sum_{i=1}^{m} D(t_0, t_i)[EL_{(A,D)}(t_i) - EL_{(A,D)}(t_{i-1})]}{\sum_{i=1}^{m} \Delta t_i D(t_0, t_{i-1})[1 - EL_{(A,D)}(t_{i-1})]} \quad (3.6)$$

Equation (3.6) is a very compact representation of fair premium S^*. It shows that once we know the expected cumulative loss on a given tranche level in question, for example $EL_{(A,D)}(t)$, the fair premium S^* is then straightforward. Unfortunately, the loss distribution function $F(x)$, which is a key element of $EL_{(A,D)}(t)$ (see equation 3.2) is not easy to derive. The main reason is the influences of the default correlation between obligors. Different default correlations may result in various shapes of loss distributions, and hence various forms of distribution functions. For example, higher defaults correlation tends to lead to a fatter tail loss distribution. Fatter tail (for example see figure 3.1, black line) means that both low level of defaults and high level of defaults are more likely to take place than the average default level. By comparison, a low default correlation (for example see figure 3.1, blue line) tends to result in a skinny tail loss distribution in that the average levels of defaults are more likely (Dominique and Julien , 2005).

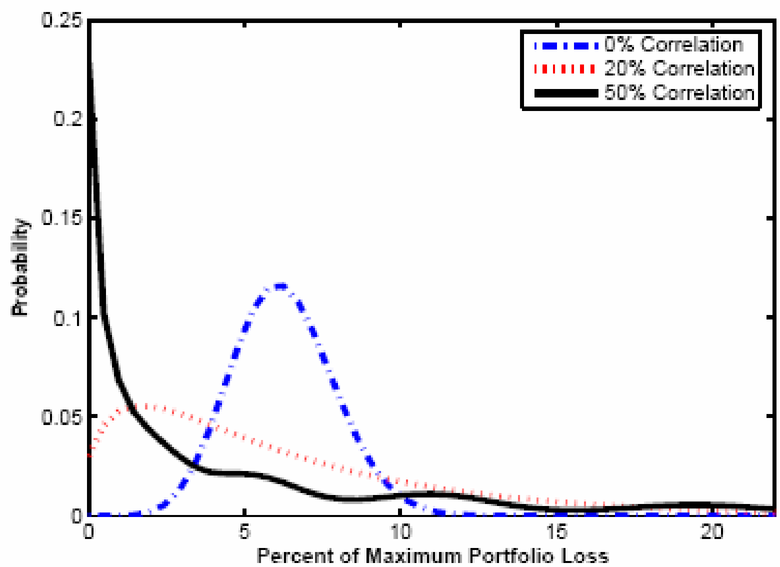


Figure 3.1 Influences of default correlations on the portfolio loss distribution

Source: Dominique and Julien 2005

Consequently, the modeling of default correlation structure plays a crucial role in calculating the loss distribution function, and hence pricing the tranche of a CDO. Note that within the CDO pricing context, we have to not only consider about the joint defaults but also the timing issue (the time to default), because premium payment depends on the outstanding notional which is reduced during the lifetime of the contract if obligors default. The purpose of the next chapter is to present the standard default correlation model and necessary theory which are essential to obtain the loss distribution in a CDO pricing context.

Default correlation

Default correlation, by definition, is the 'phenomenon that the likelihood of one obligor defaulting on its debt is affected by whether or not another obligor has defaulted on its debts'. (Douglas Lucas (2004)). For easier understanding, default correlation measures the tendency or the degree of two companies/names to default approximately simultaneously. We have positive /negative default correlation, which respectively means that if one goes to default, others are more /less easy to default.

4. Copula Function and default correlation model (factor copula model)

As is shown in the previous chapter, default correlation that determines the loss distribution has been a key element for pricing a CDO tranche. In this chapter, I would like to discuss the default correlations, which are modelled using the factor copula proposed by Li (2000). Before directly discussing the model details, we may first look through the short review of the concept and properties of a copula function, based on which the underlying principle of default correlation model/ factor copula model is built.

4.1 Definition and Basic Properties of Copula Function

Let U_1, $U_2, \ldots U_m$ be m uniform random variables; ρ be the correlation parameter.

Definition: The joint distribution function $C(u_1, u_2, \cdots u_m, \rho)$, denoted with C is called a copula function, if

$$C(u_1, u_2, \cdots u_m, \rho) = P(U_1 \le u_1, U_2 \le u_{2,\cdots} U_m \le u_m)$$

For the copula function, an important property is called Sklar's Theorem, which shows that any multivariate distribution function F can be written in the form of a copula function. In addition, the theory clearly reveals that copula may well link the marginal probability into a joint distribution.

Sklar's Theorem: If $F(x_1, x_2, \cdots x_m)$ is a joint multivariate distribution function with univariate marginal distribution functions $F(x_1)$, $F(x_2)$, $\cdots F(x_m)$, then there exists a copula function $C(u_1, u_2, \cdots u_m)$ such that

$$F(x_1, x_2, \cdots x_m) = C(F(x_1), F(x_2), \cdots F(x_m))$$

If each F_i is continuous then C is unique. Thus, copula functions provide a unifying and flexible way to study multivariate distributions.

4.2 Default correlation model (factor copula model)

A concise theoretic description of the term' default correlation' is presented in the previous section 3.2.2. Now, we start from model's point of view, there are two types of default correlation models suggested, namely reduced form models and structural models. Duffie and Singleton (1999) and Merton's (1974) already gave the specific description. Considering that the two models are quite computationally time consuming when they are used for pricing products, naturally, we come up with an idea of using the factor copula to model the default correlation. The principal underlied is to make use of the property of the copula function, which is that the factor copula created joint probability distribution for the times to default of many companies/obligors to be constructed from several marginal distributions. This default correlation factor copula model is introduced by Li (2000) and is very popular with the participants in the market. Essentially, the advantage of the copula model lies in its creation of a tractable multivariate joint distribution for a set of variables given the marginal probability distributions for the variables (Hull and White 2004). In the following, the modeling details are introduced.

Consider a portfolio of N companies/obligors and assume that the marginal probabilities of defaults are known for each company/obligor.
Define:

t_i : The time of default for the i^{th} obligor

$Q_i(t)$: The cumulative default probability function (cdf) obligor i will default before time t ; that is, $P(t_i \leq \text{t})$ i.e. the cumulative distribution function of t_i

To generate an one-factor standard copula model[6] for default time t_i , we define a latent random variables x_i $(1 \le i \le N)$

$$x_i = \rho_i Y + \sqrt{1-\rho_i^{\,2}} \cdot \varepsilon_i \qquad (4.1)$$

Where the variable x_i can be thought of as a default indicator variable for the i^{th} obligor: the lower the value of the variable, the earlier a default is likely to occur. Each x_i has two stochastic components. The first Y is a common factor which is the same for all x_i, while the second ε_i is an idiosyncratic component affecting only x_i. Both the two factors ε_i and Y have independent zero-mean unit-variance distributions. The correlation with the market is represented by ρ_i and it is in a range of [–1, 1). Since the equation has defined a correlation structure between the x_i, dependent on a single common factor Y. The correlation between x_i and x_j is $\rho_i\rho_j$.

Let us look at the general one factor copula model a bit deeper. Under the *standard* copula factor model framework: $x_i = \rho_i Y + \sqrt{1-\rho_i^{\,2}} \cdot \varepsilon_i$; if we let the ε_i's and Y's be standard normal distributions, a Gaussian copula then results. Generally, any distributions can be used for ε_i's and Y's providing they are scaled so that they have zero mean and unit variance in order to meet the requirement in the general framework. Each choice of distributions results in a different factor copula to model the default correlation structure, and thus resulting in different methods to derive loss distribution functions for pricing CDO tranche. The choice of the copula models decides the nature of the default dependence structure.

Suppose $p_i(y)$ refers to the cumulative default probability, or specifically, the conditional probability of i^{th} company that go to default before time t, then how to calculate it? How can we derive the formula from the standard one factor copula model framework?

Under the one factor copula model, x_i is mapped to t_i using a percentile-to-percentile transformation technique. This means that when x_i is small, the time t_i before default is also small. For example, the 7% point on the x_i distribution is mapped to the 7% point on the t_i distribution; and so on.

[6] Here the one-factor standard copula model is the general copula model framework we use to model the default correlation. All the other copula models including the Gaussian copula model and those introduced in the following chapters are built based on the framework.

Define

F_i: The cumulative density function of x_i i.e. cdf of x_i

H: The cumulative density function of ε_i (assuming that the ε_i is identically distributed)

Then, in general the point $x_i = x$ is mapped to $t_i = t$ where

$$x = F_i^{-1}[Q_i(t)] \equiv K \tag{4.2}$$

or equivalently

$$t = Q_i^{-1}[F_i(x)] \tag{4.3}$$

Note that in some paper, $x = F_i^{-1}[Q_i(t)]$ *is considered to be a threshold and is often given a notation* ***K*** *or* ***C*** *etc. In the rest of my paper, I use* ***K*** *by default.*

Observing the copula model shown in equation 4.1, we may find that it essentially defines a default correlation structure between the t_i's in the form of x_i, while maintaining their marginal distributions. In other words, in order to construct the default correlation structure, we do not have to define the correlation structure between the variables of interest (like time to default t_i's) directly by using the reduced form models or structural models, which greatly increase the computation complexity. With he help of the factor copula model, typically one factor copula model, we may use mapping technique to map the variables of interest (like t_i) into other more manageable variables (like x_i's) and then define a default correlation structure between those manageable variables.

Let H be the cumulative distribution function of ε_i, we can deduce that

$$\begin{aligned} P(x_i < x \mid Y = y) &= P(\rho_i Y + \sqrt{1-\rho_i^2} \cdot \varepsilon_i < x \mid Y = y) \\ &= P(\varepsilon_i < \frac{x - \rho_i Y}{\sqrt{1-\rho_i^2}} \mid Y = y) \\ &= H_i\left[\frac{F_i^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}}\right] \end{aligned} \tag{4.4}$$

Therefore, the cumulative probability of the i^{th} default by time t, conditional on the common factor y is

$$\begin{aligned} P(t_i < t \mid Y = y) = P(x_i < x \mid Y = y) &= P(\varepsilon_i < \frac{x - \rho_i Y}{\sqrt{1-\rho_i^2}} \mid Y = y) \\ &= H_i\left[\frac{F_i^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}}\right] = p_i(y) \end{aligned} \tag{4.5}$$

For simplicity, we denote $P(t_i < t \mid Y = y)$ with $p_i(y)$. Note that the conditional cumulative probability of one obligor' default $p_i(y)$ is a key input to calculate the loss distribution. The detailed derivation process of the conditional and unconditional loss distribution functions calculated from $p_i(y)$, will be presented in the next chapter in loss distribution context.

5. One factor Gaussian copula model

Given the standard default correlation models in the previous chapter, I will in the following introduce the current market benchmark model-One factor Gaussian copula mode. And based on the standard model, the procedures to derive the unconditional /conditional loss distribution functions are addressed.

5.1 One factor Gaussian copula model set up

Suppose a portfolio of reference assets consists of N obligors/companies. Recalled that Gaussian copula is actually resulted by defining the ε_i's and Y's being standard normal distributions in the **standard** copula model. In the **Gaussian** copula model, the latent variable x_i is given by

$$x_i = \rho_i Y + \sqrt{1-\rho_i^2} \cdot \varepsilon_i \qquad i = 1, 2, \cdots, N \tag{5.1}$$

Where x_i : latent random variable, following the standard normal distribution, namely, $x_i \sim$ N(0,1); $i = 1, 2, \cdots, N$

X: Gaussian vector, X= ($x_1, x_2, \ldots, x_N$), following multivariate normal distribution, namely, X~ $N_N(0, \Sigma)$

Y: common factor

ε_i: Idiosyncratic factor; $i = 1, 2, \cdots, N$

ρ_i: Correlation parameter with market, for simplicity, we may assume that $\rho_i = \rho \quad i = 1, 2, \cdots, N$

Both Y and ε_i follow the standard normal distributions

Since the default time t_i , $i = 1, 2, \cdots, N$, are modelled from the Gaussian vector X, the default times are given by $t = Q_i^{-1}[F_i(x)]$. In the case of Gaussian copula model (using the same principal we already discussed in the section 4.2)

$$t = Q_i^{-1}[\Phi_i(x)] \tag{5.2}$$

with Q_i be the cdf of t_i and Φ_i be the cumulative normal distribution function of x_i; Q_i^{-1} is just the general inverse function of Q_i. Thus x_i is given by

$$x = \Phi_i^{-1}[Q_i(t)] \equiv K \quad i = 1, 2, \cdots, N \tag{5.3}$$

The cumulative probability of the i^{th} default by time t, conditional on the common factor Y is

$$\begin{aligned} P(t_i < t \mid Y = y) &= P(x_i < x \mid Y = y) \qquad (5.4) \\ &= P\left(\varepsilon_i < \frac{x - \rho_i Y}{\sqrt{1-\rho_i^2}} \mid Y = y\right) \\ &= \Phi_i \left[\frac{\Phi^{-1}{}_i [Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}} \right] \qquad i = 1, 2, \cdots, N \\ &= p_i(y) \end{aligned}$$

5.2 Loss distribution

Before discussing the two types of loss distributions (conditional and unconditional), let us look at two assumptions for simplicity.

First, we assume that the portfolio is composed of sets of N homogeneous debt instruments, i.e.

$$\begin{aligned} &\rho_i = \rho \qquad (5.5) \\ &\Phi_i^{-1}[Q_i(t)] = \Phi^{-1}[Q_i(t)] \equiv K_i \qquad i = 1, 2, \cdots, N \\ &p_i(y) = p(y) \end{aligned}$$

This assumption ensures that all entities represented in the portfolio have the same default probability over the time period of interest.

Second, we assume that each entity represented in the portfolio corresponds to an equal share of the portfolio, or we call it equally weighted homogeneous portfolio. It guarantees the proportional relationship between the number if defaults and the percentage of default related loss. For example, in a portfolio of I reference entities, the probability of m defaults among the reference entities is equivalent to the probability of m/I % default related loss in the portfolio.

5.2.1 Conditional loss probability

Considering that the obligors x_i in the portfolio is impendent of, yet conditional on the common factor Y, and that only two outcomes are possible for obligors x_i, default or not default respectively. Therefore, we may assume the obligors follow a binomial distribution. Given the conditional cumulative probability for obligor x_i's default $p_i(y)$, we may write the conditional loss probability function *(conditional probability density function)* that totally i companies go to default by applying the binomial distribution

$$P(X = i \mid Y = y) = \binom{N}{i} p_i(y)^i (1 - p_i(y))^{N-i} \qquad (5.6)$$

$$= \binom{N}{i} p(y)^i (1 - p(y))^{N-i}$$

With

$$p(y) = P(\varepsilon_i < \frac{x - \rho Y}{\sqrt{1-\rho^2}}) = \Phi\left[\frac{\Phi^{-1}[Q_i(t)] - \rho Y}{\sqrt{1-\rho^2}}\right]$$

$$\binom{N}{i} = \frac{N!}{i!(N-i)!}$$

Note that $p_i(y)$ means the conditional probability of obligor x_i's default and it has already been derived in equation (5.4). Under the homogeneous portfolio assumption, the probability of i out of N entities which go to default is equal to the probability of the loss L_n being the amount $L_n = \frac{i}{N} \cdot A(1-R)$

5.2.2 Unconditional loss distributions

To derive the expression for the unconditional loss probability, we just need to do integration over the common factor Y

$$P(X = i) = \int_{-\infty}^{+\infty} P(X = i \mid Y = y)\phi(y)dy \qquad (5.7)$$

Where $\phi(\cdot)$ is the probability density function of standard normal distribution, because Y is following standard normal distribution.

Substituting equation 5.6 and 5.4 in equation 5.7, we obtain the following expression for the unconditional probability of i defaults in the portfolio

$$P(X = i) = \int_{-\infty}^{+\infty} \binom{N}{i} p(y)^i (1 - p(y))^{N-i} \phi(y)dy \qquad (5.8)$$

$$= \int_{-\infty}^{+\infty} \frac{N!}{i!(N-i)!} \cdot \left\{\Phi\left[\frac{\Phi^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}}\right]\right\}^i \left\{1 - \Phi\left[\frac{\Phi^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}}\right]\right\}^{N-i} \phi(y)dy$$

Therefore, the unconditional loss distribution function of defaults

$$P(X \le m) = \sum_{i=0}^{m} P(X = i) \qquad (5.9)$$

$$= \sum_{i=0}^{m} \int_{-\infty}^{+\infty} \frac{N!}{i!(N-i)!} \cdot \left\{\Phi\left[\frac{\Phi^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}}\right]\right\}^i \left\{1 - \Phi\left[\frac{\Phi^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}}\right]\right\}^{N-i} \phi(y)dy$$

Note that for Gaussian copula model, by assumption, $\rho_i = \rho$ holds for all the obligors.

5.3 The large portfolio approximation for the one factor model

The calculation of the unconditional loss distribution in equation (5.9) is computationally trivial, especially for a very large N. In order to overcome this problem, Vasicek (1987) proposed the large portfolio approximation approach, which is a convenient and efficient way for approximation when the N tends to be ∞.

Let x be the fraction of defaulted entities in the portfolio, then

$$F_N(x) = p(X \le x) \tag{5.10}$$

$$= \lim_{N\to\infty} \sum_{i=0}^{Nx} \int_{-\infty}^{+\infty} \frac{N!}{i!(N-i)!} \cdot \left\{ \Phi\left[\frac{\Phi^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}} \right] \right\}^i \left\{ 1 - \Phi\left[\frac{\Phi^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}} \right] \right\}^{N-i} \phi(y)dy$$

Define $\Phi^{-1}[Q_i(t)] = K_i$, and $\Phi\left[\frac{K_i - \rho_i Y}{\sqrt{1-\rho_i^2}} \right] = m_i$ with $\rho_i = \rho$ holds for all the obligors, we have

$$Y = \frac{\sqrt{1-\rho}\,\Phi^{-1}(m_i) - K_i}{\sqrt{\rho}} \tag{5.11}$$

Hence

$$F_N(x) = p(X \le x) \tag{5.12}$$

$$= \lim_{N\to\infty} \sum_{i=0}^{Nx} \int_{-\infty}^{+\infty} \frac{N!}{i!(N-i)!} \cdot \{m_i\}^i \{1 - m_i\}^{N-i} \, d\Phi\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}(m_i) - K_i}{\sqrt{\rho}}\right]$$

Since

$$\lim_{N\to\infty} \sum_{i=0}^{Nx} \int_{-\infty}^{+\infty} \frac{N!}{i!(N-i)!} \cdot \{m_i\}^i \{1 - m_i\}^{N-i} = \begin{cases} 0 & \text{if x} \le m_i \\ 1 & \text{otherwise} \end{cases} \tag{5.13}$$

The cumulative distribution function of loss of large portfolio is given by

$$F(x) = p(X \le x)$$

$$= \Phi\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}(x) - K_i}{\sqrt{\rho}}\right] \tag{5.14}$$

It is easy to see the expression of loss distribution function $F(x)$ using large portfolio approach displayed in equation (5.14) is much compacter and handier than the one shown in equation (5.9). With the above expression, we may easily calculate the expected tranche losses and then the corresponding CDO tranche premium.

5.4 Evaluation of the Gaussian copula approach

Gaussian factor copula model has been widespread applied for the valuation of types of instruments and already become a current industry standard. Its attractiveness of pricing CDOs is evident due to simplicity and straightforward application, which merely requires simple and fast numerical integration techniques (Guegan, Dominique and Houdain, Julien, 2005). However, it is argued that the assumptions for this model are too strong. For example, the model assumes the correlation parameter with market (compound implied correlation) $\rho_i = \rho$ holds for all the obligors, recovery rate is also constant etc. Recently many researches reveal that several serious drawbacks do exist in Gaussian copula model, which tend to result in mispricing the tranche spreads. Thus it might not be an ideal method to price the tranche of CDO. In the following, two categories of the shortcomings are discussed.

5.4.1 Correlation structure

Under the Gaussian copula model, a flat correlation structure is drawn due to the constant correlation parameter ρ_i assumption. But a flat correlation structure is not sufficient to reflex the heterogeneity of the underlying assets (e.g. x_i). Accordingly, since one single number $\rho\rho$ couldn't explain well the complex relationship between the default times of different assets like $x_i x_j$ thus it is obviously not appropriate to use Gaussian copula model, especially its constant implied compound correlations ρ of traded tranches.

Implied correlation

Mashal et al. (2004) define the implied correlation (the correlation parameter ρ_i) of a tranche as *the uniform asset correlation number that makes the fair or theoretical value of a tranche equal to its market quote*. In other words, Hull and White (2004) for example define the implied correlation for a tranche as *the correlation that causes the value of the tranche to be zero*. Implied correlations do exist within each tranche yet generally not uniquely defined. In addition, tranche spreads are not necessarily monotone in correlation. Therefore, we may observe the market prices that may not be attainable by just one choice of constant correlation parameter as Gaussian copula does (Svenja Hager and Rainer Schöbel, 2005).

Tranche	**[0,3]**	**[3,6]**	**[6,9]**	**[9,12]**
Market quote	**27.6%**	**1.68%**	**0.7%**	**0.43%**
Gauss ρ=21.9%	27.6%	2.95%	1.05%	0.42%
Gauss ρ=4.2%	43.1%	1.68%	0.1%	0.005%
Gauss ρ=87.9%	-------	1.68%	1.35%	1.14%
Gauss ρ 14.8%	33.2%	2.68%	0.7%	0.2%
Gauss ρ=22.3%	27.3%	2.96%	1.07%	0.43%
Gauss ρ=30.5%	21.6%	3.05%	1.35%	0.67%

Table 5.1 Implied correlations from Gaussian copula

The table 5.1 displays the implied correlations parameters from the Gaussian copula, and we may observe the ρ does exist in every tranche and also not unique, such as in the tranche **[3, 6],** ρ=4.2% and ρ=87.9% result in the same premium 1.68%.

5.4.2 Correlation smile

As the market quotes on CDOs become more readily available, researchers are able to calibrate their model parameters to those real market quotes. Under the chosen model, using the correlation matrix, we may compute the spreads for each traches. And from these spreads the implied correlation can be derived. In the real market, quotations available indicate that different tranches on the same underlying portfolio trade at different implied correlations, which in the following figure 5.1 resembles a smile skew, called correlation smile. However, it is not the case under the standard Gaussian model, which expects the compound correlation being equal for every single tranche and thus the Gaussian copula model doesn't lead to a smile skew yet a flat structure. From the table 5.1, we may also observe the implied correlations for each tranche are not the same. Such problems are mainly due to the simplifying assumptions that the correlations as well as the recovery rates and CDS spreads are constant and equal for all obligors.

The correlation smile in Figure 5.1 plot below shows a lower default time correlation on the mezzanine tranche than on the equity and senior tranches. So, we can conclude that the degree of default clustering assumed by the market appears to be higher for the equity and senior tranches. Quotations available in the market indicate that different tranches on the same underlying portfolio trade at different implied correlations. It might be that the Gaussian copula model could not accurately reflect the joint distribution of default times.

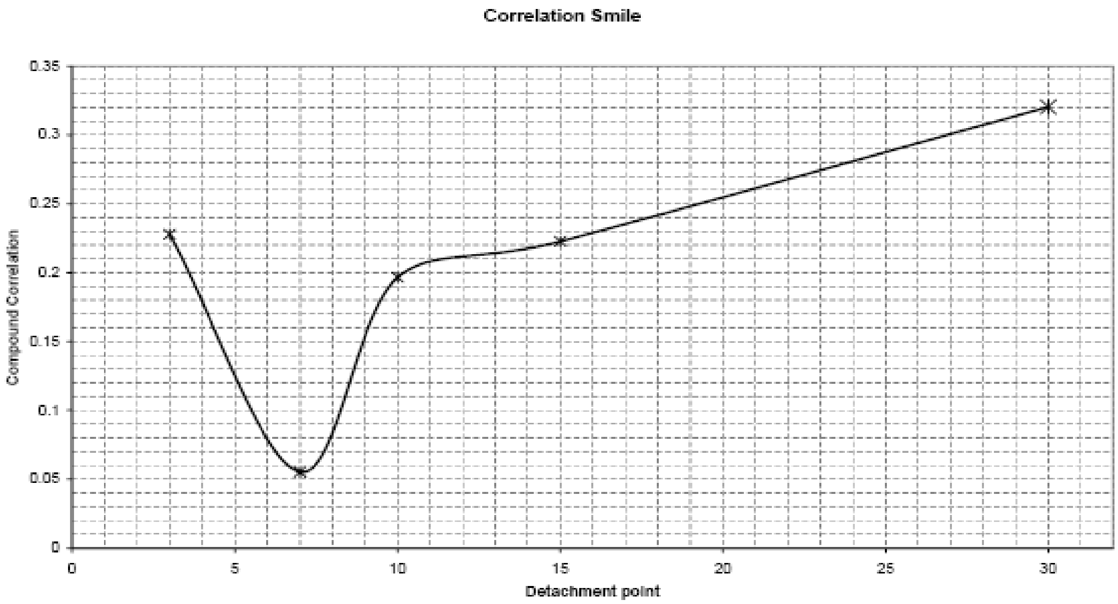


Figure5.1: Compound Correlations plotted against the detachment level of the tranches. Quotes on standard tranches on the DJ CDX basket are used from the 18th of October 2004. (Source: Van der Voort, M., 2005)

Different explanations have been searched for the correlation smile. One of the explanations could be that the different groups of investors (protection sellers i.e. hedge funds for equity tranches; banks and security firms for mezzanine tranches) hold different views about the correlations across tranches. Another possible explanation is that the uncertainty of how to select the optimal model for valuing credit risk correlations in the view of the market participants might be as well reflected in the correlation smile. Although the index tranche market has grown over the last years, prices are very like to be influenced by the local conditions. Moreover, supply and demand imbalances on the market might also induce the shape of the correlation smile, because for example the mezzanine tranches are extremely popular among investors.

In sum, using the standard market model with only single asset correlation parameter is very likely to result in a mispricing problem when valuing premiums of CDOs. Considering the existence of such problems and drawbacks, many researchers are working on the extensions of the standard Gaussian copula models, trying to produce models that may duplicate the smile skew and fit the market quotes better.

6. Extensions to the Gaussian model and its comparisons

There has been much interest recently in simple extensions of the Gaussian one-factor model in order to match the "correlation smile" in the CDO market. Gregory and Laurent (2004) propose a correlation structure built from groups specifying intra and inter group correlation coefficients and they introduce some dependence between recovery rates and defaults. Hull and White (2005) recommend the use of a double Student-t one-factor model. Andersen and Sidenius (2005) introduce random recovery rates and random factor loadings in the model. Burtschell, Gregory and Laurent (2005) propose a comparative analysis of the previous CDO pricing models and illustrate the fact that these models should be improved. In the following, I will review some typical extensive models in details.

6.1 Student t copula model

In student t approach, the vector X= ($x_1, x_2, \ldots, x_N$) follows a student t distribution with u degrees of freedom. For simplicity, we consider the symmetric situation

$$x_i = \rho_i \cdot \sqrt{w} \cdot Y + \sqrt{1-\rho_i^2} \cdot \sqrt{w} \cdot \varepsilon_i \qquad (6.1)$$
$$= \sqrt{w}(\rho_i Y + \sqrt{1-\rho_i^2} \cdot \varepsilon_i) \qquad i = 1, 2, \cdots, N$$

Where Y, ε_i : independent Gaussian random variables

W: has inverse Gamma distribution with the parameter u/2 (or $\frac{u}{w} \sim \chi_n^2$) and w is independent of x_i

Cov(x_i, x_j)=($\frac{u}{u-2}$) $\rho_i \rho_j$ (u>2) (6.2)

We denote by $\tilde{T}$ the cdf of standard univariate student t distribution, thus the default time t is given by

$$t_i = Q_i^{-1}[\tilde{T}_i(x_i)] \qquad (6.3)$$

Equivalently

$$x_i = \tilde{T}_i^{-1}[Q_i(t_i)] \equiv K_i \qquad (6.4)$$

Accordingly, the cumulative probability of the i^{th} obligor's default by time t, conditional on the common factor Y is

$$P_i(y) = P(t_i < t \mid Y = y)$$
$$= P(x_i < x \mid Y = y)$$
$$= P(\rho_i Y \sqrt{W} + \sqrt{1-\rho_i^2} \sqrt{W} \cdot \varepsilon_i < K_i \mid Y = y)$$

$$= P(\varepsilon_i < \frac{K_i - \rho_i Y\sqrt{W}}{\sqrt{1-\rho_i^2}\sqrt{W}})$$

$$= \Phi(\frac{K_i - \rho_i Y\sqrt{W}}{\sqrt{1-\rho_i^2}\sqrt{W}}) \qquad (6.5)$$

6.2 Double t copula model

Define latent random vector X= ($x_1, x_2, \ldots, x_N$), which modelled the default times $t_i,\ i = 1, 2, \cdots, N$

$$x_i = (\frac{u-2}{u})^{1/2}\rho_i Y + \sqrt{1-\rho_i^2}(\frac{\bar{u}-2}{\bar{u}})\cdot\varepsilon_i \qquad (6.6)$$

Where Y, ε_i: both follow student t distribution with degree of freedom u and $\bar{u}$ respectively

ρ_i: $\rho_i \geq 0$

It is noted that student t distributions are not stable under convolution, though two main factors Y, ε_i both follow t-distribution, x_i didn't. Thus the copula associates with ($x_1, x_2, \ldots, x_N$) is not a student copula , which distinguish itself with student t copula model.

Simply, default time is given by

$$t_i = Q_i^{-1}[F_i(x_i)]$$

Equivalently

$$x_i = F_i^{-1}[Q_i(t_i)] \equiv K_i \qquad (6.7)$$

Where Q_i: the cumulative distribution function of t_i

F_i: the cumulative distribution function of x_i

Accordingly, the cumulative probability of the i^{th} obligor's default by time t, conditional on the common factor Y is

$$P_i(y) = P(t_i < t \mid Y = y) \qquad (6.8)$$

$$= P(x_i < x \mid Y = y)$$

$$= P((\frac{u-2}{u})^{1/2}\rho_i Y + \sqrt{1-\rho_i^2}(\frac{\bar{u}-2}{\bar{u}})\cdot\varepsilon_i < K_i \mid Y = y)$$

$$= P(\varepsilon_i < \frac{K_i - (\frac{u-2}{u})^{1/2}\rho_i Y}{\sqrt{1-\rho_i^2}(\frac{\bar{u}-2}{\bar{u}})})$$

$$= \tilde{T}_i\left(\frac{K_i - \left(\frac{u-2}{u}\right)^{1/2} \rho_i Y}{\sqrt{1-\rho_i^2 \left(\frac{\bar{u}-2}{\bar{u}}\right)}}\right)$$

6.3 Clayton copula model

Define a random variable L, following a standard Gamma distribution with shape parameter $\frac{1}{\theta}$ where $\theta > 0$, i.e. $L \sim \Gamma(\frac{1}{\theta})$.

The probability of density function of L is given by

$$f(x) = \frac{1}{\Gamma(\frac{1}{\theta})} \mathrm{e}^{-x} x^{(1-\theta)/\theta} \quad ; \quad x > 0 \tag{6.9}$$

We denote by ψ the laplace transformation of L

$$\psi(y) = \int_0^{+\infty} f(x) e^{-yx} dx = (1+y)^{-1/\theta} \tag{6.10}$$

Let $u_1, u_2, \cdots u_N$ be independent uniform random variables, and independent of L

The Clayton factor model is written as

$$x_i = \psi\left(-\frac{\ln u_i}{L}\right) \tag{6.11}$$

Then the default time t_i are given by

$$t_i = Q_i^{-1}(x_i)$$

Or equivalently

$$x_i = Q_i(t_i) \tag{6.12}$$

The cumulative probability of the i^{th} obligor's default by time t, conditional on the common factor Y is

$$P_i(l) = P(x_i < x \mid L = l) \tag{6.13}$$

$$= P\left(\psi\left(-\frac{\ln u_i}{L}\right) < Q_i(t) \mid L = l\right)$$

$$= \exp(l \cdot (1 - Q_i(t)^{-\theta}))$$

Gregory & Laurent [2003] and Laurent & Gregory [2003] have been considering this model in a credit risk context.

6.4 Normal inverse Gaussian model

Normal inverse Gaussian distribution (NIG) is a special case of the group of generalized hyperbolic distributions (Barndorff-Nielsen). They are stable under convolution in certain conditions and the cumulative density function (cdf), probability density function and inverse distribution function can still be computed sufficiently fast. Kalemanova et al has applied the normal inverse Gaussian models to the CDO pricing recently and proved a good fit to market data.

6.4.1 Definition and properties of the NIG distribution

The normal inverse Gaussian distribution is a mixture of normal and inverse Gaussian distributions.

Definition: a Non-negative random variable Y has inverse Gaussian (IG) distribution with parameters $\alpha > 0$ and $\beta > 0$, i.e. $y \sim IG(\alpha, \beta, y)$, if its density function satisfy

$$f_{IG}(y;\alpha,\beta) = \begin{cases} \frac{\alpha}{\sqrt{2\pi\beta}} y^{-3/2} \exp(\frac{-(\alpha-\beta y)^2}{2\beta y}) & \text{if y>0} \\ 0 & \text{otherwise} \end{cases} \quad (6.14)$$

Definition: A random variable X follows a normal inverse Gaussian (NIG) distribution with parameters α, β, η, γ, i.e. $X \sim NIG(\alpha,\beta,\eta,\gamma,x)$, if

$$X|(Y=y) \sim N(\eta + \beta y, y) \quad (6.15)$$

and $Y \sim IG(\gamma\lambda, \lambda^2)$ with $\lambda := \sqrt{\alpha^2 - \beta^2}$

satisfy constraints

$$0 \le |\beta| < \alpha \text{ and } \eta > 0$$

Denote by $f_{NIG}(x,\alpha,\beta,\eta,\gamma)$, $F_{NIG}(x,\alpha,\beta,\eta,\gamma)$ the density function and the cumulative distribution function of $X \sim NIG(\alpha,\beta,\eta,\gamma,x)$ respectively. We have

$$f_{NIG}(x,\alpha,\beta,\eta,\gamma) = \frac{\gamma\lambda \exp(\gamma\lambda + \beta(x-\eta))}{\pi\sqrt{\gamma^2 - (x-\eta)^2}} \cdot K_1(\alpha\sqrt{\gamma^2 - (x-\eta)^2}) \quad (6.16)$$

where $K_1(w) := \frac{1}{2}\int_0^\infty \exp(-\frac{1}{2}w(t+t^{-1}))dt$ is the modified Bessel function of the third Kind. The density function relies on four parameters α, β, η, γ, with α related to steepness, β influencing the symmetry, and η, γ respectively to location and scale.

Next I will introduce two important properties of NIG distribution. One is called **"scaling property"**, which is given by

$$X \sim NIG(\alpha,\beta,\eta,\gamma,x) \Rightarrow cX \sim NIG(\frac{\alpha}{c},\frac{\beta}{c},c\eta,c\gamma) \quad (6.17)$$

where c is a constant.

The second property is about NIG distribution's **stability** under the convolution situation. Let X, Y, T be independent random variables

$X \sim NIG(\alpha,\beta,\eta_1,\gamma_1,x)$; $Y \sim NIG(\alpha,\beta,\eta_2,\gamma_2,y)$

$$\Rightarrow T = X + Y \sim NIG(\alpha,\beta,\eta_1+\eta_2,\gamma_2+\gamma_1,t) \tag{6.18}$$

The mean and variance of a random variable $X \sim NIG(\alpha,\beta,\eta,\gamma,x)$ are given by

$$E(x) = \eta + \gamma \frac{\beta}{\sqrt{\alpha^2-\beta^2}}$$

$$Var(x) = \frac{\gamma\alpha^2}{\sqrt{\alpha^2-\beta^2}}$$

$$Skew(x) = 3(\frac{\beta}{\alpha})(\frac{1}{\sqrt{\eta\gamma}})$$

$$Kurt(x) = 3[1+4(\frac{\beta}{\alpha})^2](\frac{1}{\eta\gamma})$$

6.4.2 Normal inverse model set up

$$x_i = \rho_i Y + \sqrt{1-\rho_i^{\,2}}\cdot\varepsilon_i$$

where ρ_i : constant correlation parameter, $i = 1,2,...N$

ε_i,Y: independent NIG random variables satisfy

$$Y \sim NIG(\alpha,\beta,\frac{-\alpha\beta}{\sqrt{\alpha^2-\beta^2}},\alpha)$$

$$\varepsilon_i \sim NIG(\frac{\sqrt{1-\rho_i^{\,2}}}{\rho_i},\alpha,\frac{\sqrt{1-\rho_i^{\,2}}}{\rho_i}\beta,\frac{-\sqrt{1-\rho_i^{\,2}}}{\rho_i}\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}},\frac{\sqrt{1-\rho_i^{\,2}}}{\rho_i}\alpha)$$

Note that the NIG implied common factor Y is different from Gaussian common factor[7].

Using the two properties we discussed in 6.5.1, we get

$$x_i \sim NIG(\frac{\alpha}{\rho_i},\frac{\beta}{\rho_i},-\frac{1}{\rho_i}\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}},\frac{\alpha}{\rho_i}) \tag{6.19}$$

For simplicity, we denote probability distribution function of x_i, $i = 1,2,...N$ $F_{NIG}(x,\frac{\alpha}{\rho_i},\frac{\beta}{\rho_i},-\frac{1}{\rho_i}\frac{\alpha\beta}{\sqrt{\alpha^2-\beta^2}},\frac{\alpha}{\rho_i})$ with $F_{NIG(\frac{1}{\rho_i})}(x)$. Hence, ε_i has distribution function of $F_{NIG(\frac{\sqrt{1-\rho_i^{2}}}{\rho_i})}(x)$.

The default time is given by

[7] For a more detailed description, see Guegan, Dominique and Houdain, Julien, 2005.

$$t_i = Q_i^{-1}[F_{NIG(\frac{1}{\rho_i})}(x_i)]$$

equivalently

$$x = F^{-1}_{NIG(\frac{1}{\rho_i})}[Q_i(t)] \equiv K_i \qquad (6.19)$$

then the conditional probability of the i^{th} obligor/company that defaults before time t is given by

$$\begin{aligned} p(t_i < t \mid Y) &= p_i(y) \\ &= p(\rho_i Y + \sqrt{1-\rho_i^2} < K_i \mid Y = y) \\ &= p(\varepsilon_i < \frac{K_i - \rho_i Y}{\sqrt{1-\rho_i^2}}) \\ &= F_{NIG(\frac{\sqrt{1-\rho_i^2}}{\rho_i})}(\frac{K_i - \rho_i Y}{\sqrt{1-\rho_i^2}}) \end{aligned} \qquad (6.20)$$

The large portfolio approximation is given by (according to the section 5.3)

$$F_\infty(x) = P(X \le x) = F_{NIG(1)}[\frac{(\sqrt{1-\rho_i^2})F^{-1}_{NIG(\frac{\sqrt{1-\rho_i^2}}{\rho_i})}(x) - K_i}{\rho_i}] \qquad (6.21)$$

6.5 Stochastic correlation Gaussian models

Define a latent random variable $x_i, \quad i = 1, 2, \cdots, N$

$$\begin{aligned} x_i &= B_i(\rho_i Y + \sqrt{1-\rho_i^2} \cdot \varepsilon_i) + (1-B_i)(\overline{\rho}_i Y + \sqrt{1-\overline{\rho}_i^2} \cdot \varepsilon_i) \\ &= [B_i\rho_i + (1-B_i)\overline{\rho}_i]Y + \sqrt{1-[B_i\rho_i + (1-B_i)\overline{\rho}_i]^2} \cdot \varepsilon_i \end{aligned} \qquad (6.22)$$

Where B_i: Bernoulli random variables,

Y, ε_i: standard Gaussian random variables, and independent of each other and also B_i

$\rho_i, \overline{\rho}_i$: correlation parameters with market, $0 \le \overline{\rho}_i \le \rho_i \le 1$

In the above stochastic correlation Gaussian model, it is a convex sum of one factor Gaussian copulas, involving a mixing distribution over factor exposures $\rho_i, \overline{\rho}_i$. In our case, there are here two states for each obligor/name, one corresponding to a high correlation and the other to a low correlation (Burtschell, Gregory and Laurent (2005)).We denote by $p = Ber(B_i = 1)$ then $1 - p = Ber(B_i = 0)$. Hence, we have a factor exposure ρ_i with probability p and $\overline{\rho}_i$ with probability

(1- *p*). In addition, the marginal distributions of x_i are Gaussians.
The default time is given by

$$t_i = Q_i^{-1}[\Phi_i(x_i)]$$

Note that the default times are independent conditionally on Y, then, the conditional cumulative default probabilities is

$$P_i(y) = P(x_i < x \mid Y = y) \tag{6.23}$$

$$= p \cdot \Phi\left[\frac{\Phi^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}}\right] + (1-p)\cdot\Phi\left[\frac{\Phi^{-1}[Q_i(t)] - \bar{\rho}_i Y}{\sqrt{1-\bar{\rho}_i^2}}\right]$$

6.6 Random factor loading (RFL) model

The random factor loading model is introduced by Andersen and Sidenius**.** Its underlying idea is to make factor loadings, basing on the factor models, being functions of the system/common factors themselves. Interpreting the systematic factor as the 'state of market' with its value being high in good time and lower in the bad time, the RFL model may mimic the well-known empirical effect that equity (and thereby asset) correlations are higher in a bear market than in a bull market (Andersen, L. and Sidenius, J. (2004))

6.6.1 Model set up

$$x_i = \rho_i(Y)Y + v_i \cdot \varepsilon_i + m_i \tag{6.24}$$

Where ρ_i: factor loading, with d-dimension

Y, ε_i: Y is a d-dimension variable i.e. Y = $(Y_1, \ldots, Y_d)$, $i = 1, 2, \cdots, N$
Both are independent random variables with zero mean and unit variance

v_i, m_i: they are set so that x_i has zero mean and unit variance distribution

Let F^Y be the cdf of Y, then v_i, m_i are given by

$$v_i = \sqrt{1 - Var(\rho_i(Y)Y)} = 1 - \int_{R^d} (\rho_i(Y)Y)^2 dF^Y(Y) + m_i \tag{6.25}$$

$$m_i = -E[\rho_i(Y)Y] = -\int_{R^d} \rho_i(Y)Y dF^Y(Y) \tag{6.26}$$

The default time t_i is given by

$$t_i = Q_i^{-1}[F_i^x(x_i)]$$

Equivalently, we get

$$x_i = F_i^{-x}[Q_i(t_i)] = K_i \tag{6.27}$$

Where F_i^x is distribution function of x_i, Q_i is t_i's, K_i is a threshold value.

Then the individual conditional default probability (default by time t, conditional on Y) is given by

$$\begin{aligned} p_i(y) &= p(t_i < t \mid Y = y) \\ &= p(x_i < K_i \mid Y = y) \\ &= p(x_i < x \mid Y = y) \\ &= F_i^{\varepsilon}(\frac{F_i^{-x}[Q_i(t_i)] - \rho_i(Y)Y - m_i}{v_i}) \end{aligned} \tag{6.28}$$

Unconditional probability of the i^{th} obligor that defaults before time t is

$$\begin{aligned} p(t_i < t) &= p(x_i < x) \\ &= E[\,p(\varepsilon_i < \frac{F_i^{-x}[Q_i(t_i)] - \rho_i(Y)Y - m_i}{v_i} \mid Y)] \\ &= \int_{R^d} F_i^{\varepsilon}(\frac{F_i^{-x}[Q_i(t_i)] - \rho_i(Y)Y - m_i}{v_i})dF^Y(Y) \end{aligned} \tag{6.29}$$

Note that it is advantageous in practice, to deal with $\rho_i(Y)$ in 'separable structure', which means that

$$\rho_i(Y) = [\rho_{i,1}(Y_1),...,\rho_{i,d}(Y_d)]$$

Its merit reflect in the simplification of the mean and variance computation, for instance

$$Var(\rho_i(Y)Y) = \sum_{j=1}^{d} Var(\rho_{i,j}(Y_j)Y_j)$$

$$Cov(\rho_i(Y)\cdot\rho_j(Y)\cdot Y) = \sum_{s=1}^{d} Cov(\rho_{i,s}(Y_s)\cdot\rho_{j,s}(Y_s)\cdot Y_s)$$

6.6.2 Gaussian copula with RFL

As is shown in section 6.6.1, equation 6.24 is the general RFL model differing in the functional relationship between systematic factors (Y_j) and loading (ρ_i), as well as in the choice of distribution for factors and residuals.

For illustration, we consider a specific example, building on the Gaussian copula model. Specifically,

$$x_i = \rho_i(Y)Y + v_i \cdot \varepsilon_i + m_i \tag{6.30}$$

Where ε_i, Y_j: both follow the standard normal distribution, and are independent of each other , j=1,2,...,d, $i = 1, 2, \cdots, N$

$\rho_{i,j}$: Often written as ρ_i for simplicity and it is the factor loading, N*d matrix. We define it to be a two point distribution, which is given by

$$\rho_{i,j}(Y_j) = \begin{cases} a_{ij} & \text{if } Y_j \le \theta_{ij} \\ b_{ij} & \text{otherwise} \end{cases}$$

With a_{ij} and b_{ij} , both matrix being positive constant, $\theta_{ij} \in R$

v_i, m_i: the same formula as the general one

For better understanding, we may think of that as a regime-switching model where the loading takes the value a_{ij} with probability $\Phi(\theta_{ij})$, and value b_{ij} with a probability [1- $\Phi(\theta_{ij})$]. If a_{ij} > b_{ij} , the factor loadings decrease in Y_j and thus, intuitively, the asset values couple more strongly to "the economy" in bad times than in good times.

By using the Gaussian copula with RFL model, we may not only crudely mimic an empirical dependence of correlation on the broad market condition, but also generate a base correlation skew when a_{ij} > b_{ij}. To elaborate, consider in the view of senior tranche investor. This investor will only experience losses on his position when several names/companies default together (or consider it to be extreme loss outcomes---both very high and very low levels of defaults). Generally, high values of correlation parameters/factor loadings tend to result in fatter tails of distribution, which mean the extreme loss outcomes. In our case, the systematic factor Y should be low, and then the factor loadings will be high, making it appear to senior investor that correlation is high. For the equity tranche holder on the other hand, who is likely to bear losses even in scenarios where systematic factor Y is not low, the effective factor loading will appear as a weighted average between a_{ij} and b_{ij} . Hence the world will thus look as if correlations are of average magnitude to them. Evidently, if a_{ij} = b_{ij}, it is back to the constant factor loading of Gaussian case.

Calculation of v_i and m_i

In the Gaussian copula with RFL model: v_i, m_i are set so that x_i has zero mean and unit variance distribution.

$$m_i = -E[\rho_i(Y)Y] = -\sum_{j=1}^{d}[-a_{ij}\varphi(\theta_{ij}) + b_{ij}\varphi(\theta_{ij})] \qquad (6.31)$$

$$v_i = \sqrt{1 - Var(\rho_i(Y)Y)} = \sqrt{1 - \sum_{j=1}^{d} V_{ij}} \qquad (6.32)$$

Where $V_{ij} = a_{ij}^2[\Phi(\theta_{ij}) - \theta_{ij}\varphi(\theta_{ij})] + b_{ij}^2[\theta_{ij}\varphi(\theta_{ij}) + 1 - \Phi(\theta_{ij})] - [-a_{ij}\varphi(\theta_{ij}) + b_{ij}\varphi(\theta_{ij})]^2$ and I denote $Var(\rho_{i,j}(Y_j)Y_j$) with V_{ij} (we assume the separable structure).

Proof: using the lemma below, we may get

$$E[\rho_{i,j}(Y_j)Y_j] = E(a_{ij}1_{Y_j \le \theta_{ij}} Y_j + b_{ij}1_{Y_j > \theta_{ij}} Y_j)$$
$$= -a_{ij}\varphi(\theta_{ij}) + b_{ij}\varphi(\theta_{ij})$$
$$E[\rho_{i,j}(Y_j)^2 Y_j^2] = E(a_{ij}^2 1_{Y_j \le \theta_{ij}} Y_j^2 + b_{ij}^2 1_{Y_j > \theta_{ij}} Y_j^2)$$
$$= a_{ij}^2[\Phi(\theta_{ij}) - \theta_{ij}\varphi(\theta_{ij})] + b_{ij}^2[\theta_{ij}\varphi(\theta_{ij}) + 1 - \Phi(\theta_{ij})]$$

Then

$$Var(\rho_{i,j}(Y_j)Y_j) = E[\rho_{i,j}(Y_j)^2 Y_j^2] - \{E[\rho_{i,j}(Y_j)Y_j]\}^2$$

From the 'separable structure' method,

$$Var(\rho_i(Y)Y) = \sum_{j=1}^{d} Var(\rho_{i,j}(Y_j)Y_j)$$

Lemma: for a standard Gaussian variable x and arbitrary constants a and b, we have

$$E(1_{a<x\le b}\, x) = 1_{b>a}(\varphi(a) - \varphi(b))$$
$$E(1_{a<x\le b}\, x^2) = 1_{b>a}(\Phi(b) - \Phi(a)) + 1_{b>a}(a\varphi(a) - b\varphi(b))$$

Particularly,

$$E(1_{x\le b}\, x) = -\varphi(b)$$
$$E(1_{x>a}\, x^2) = a\varphi(a) + [1 - \Phi(a)]$$
$$E(1_{x\le b}\, x^2) = \Phi(b) - b\varphi(b)$$
$$E(1_{x>a}\, x) = \varphi(a)$$

Individual conditional and unconditional default probability

Assume that the dimension of systematic factor Y d=1, then the individual conditional default probability is given by

$$p(t_i \le t / Y) = p(x_i \le K_i / Y)$$
$$= p(\varepsilon_i \le \frac{K_i - \rho_i(Y)Y - m_i}{v_i}) \tag{6.33}$$
$$= \Phi(\frac{K_i - (a_i 1_{Y\le\theta_i} Y + b_i 1_{Y>\theta_i} Y) - m_i}{v_i})$$

Similarly, the unconditional probability of i^{th} obligor that defaults before time t is (the same principle presented in the section 5.3 within Gaussian copula model context, which is to calculate the integrals over the common factor Y)

$$p(t_i \le t) = p(x_i \le K_i) \tag{6.34}$$

$$= \Phi_2(\frac{K_i - m_i}{\sqrt{v_i^2 + a_i^2}}, \theta_i; \frac{a_i}{\sqrt{v_i^2 + a_i^2}}) + \Phi(\frac{K_i - m_i}{\sqrt{v_i^2 + b_i^2}}) - \Phi_2(\frac{K_i - m_i}{\sqrt{v_i^2 + b_i^2}}, \theta_i; \frac{b_i}{\sqrt{v_i^2 + b_i^2}})$$

Proof: By using the notation and lemma below, we may get

$$p(t_i \le t) = p(a_i 1_{Y<\theta_i} Y + b_i 1_{Y>\theta_i} Y + \varepsilon_i v_i + m_i \le K_i)$$
$$= E[\, p(\varepsilon_i \le \frac{K_i - (a_i 1_{Y<\theta_i} Y + b_i 1_{Y>\theta_i} Y) - m_i}{v_i} \,|\, Y)]$$
$$= E[\Phi(\frac{K_i - (a_i 1_{Y<\theta_i} Y + b_i 1_{Y>\theta_i} Y) - m_i}{v_i})]$$
$$= \int_{-\infty}^{\theta_i} \Phi(\frac{K_i - a_i Y - m_i}{v_i}) \varphi(Y) dY + \int_{\theta_i}^{\infty} \Phi(\frac{K_i - b_i Y - m_i}{v_i}) \varphi(Y) dY$$

and

$$\int_{\theta_i}^{\infty} \Phi(\frac{K_i - b_i Y - m_i}{v_i}) \varphi(Y) dY = \int_{-\infty}^{\infty} \Phi(\frac{K_i - b_i Y - m_i}{v_i}) \varphi(Y) dY - \int_{-\infty}^{\theta_i} \Phi(\frac{K_i - b_i Y - m_i}{v_i}) \varphi(Y) dY$$

Notation: Φ, Φ_2, φ denote respectively the standard Gaussian density function, the standard Gaussian cumulative distribution function and the standard bivariate Gaussian cumulative distribution function. And the K_i is a threshold value.

Lemma: For arbitrary real constants *a*, *b* and *c*

$$\int_{-\infty}^{\infty} \Phi(ax + b) \varphi(x) dx = \Phi(\frac{b}{\sqrt{1 + a^2}})$$
$$\int_{-\infty}^{\infty} \Phi(ax + b)^2 \varphi(x) dx = \Phi_2(\frac{b}{\sqrt{1 + a^2}}, \frac{b}{\sqrt{1 + a^2}}; \frac{a^2}{\sqrt{1 + a^2}})$$
$$\int_{-\infty}^{c} \Phi(ax + b) \varphi(x) dx = \Phi_2(\frac{b}{\sqrt{1 + a^2}}, c; \frac{-a}{\sqrt{1 + a^2}})$$

6.6.3 The large portfolio approximation

Consider the individual conditional default probability in Equation …

$$p_i(y) = \Phi(\frac{K_i - (a_i 1_{Y<\theta_i} Y + b_i 1_{Y>\theta_i} Y) - m_i}{v_i}) \qquad (6.35)$$

Let the loss X be the fraction of the defaulted entities in the portfolio and it is given by $p_i(y)$. Then the large portfolio approximation is given by

$$\lim_{N\to\infty} 1 - F_N(x) = \lim_{N\to\infty} p(X \geq x)$$
$$= P(p_i(y) \geq x)$$
$$= P[\frac{K_i - \rho_i(Y)Y - m_i}{v_i} \geq \Phi^{-1}(x)] \quad (6.36)$$
$$= P(\rho_i(Y)Y \leq K_i - v_i\Phi^{-1}(x) - m_i))$$

For simplicity, we denote $\Omega(x) \equiv K_i - v_i\Phi^{-1}(x) - m_i$, then

$$\lim_{N\to\infty} p(X \geq x)$$
$$= P[\rho_i(Y)Y \leq \Omega(x), Y \leq \theta_i] + P[\rho_i(Y)Y \leq \Omega(x), Y > \theta_i]$$
$$= P[a_iY \leq \Omega(x), Y \leq \theta_i] + P[b_iY \leq \Omega(x), Y > \theta_i]$$
$$= 1 - \left(\Phi[\min(\frac{\Omega(x)}{a_i}, \theta_i)] + 1_{\frac{\Omega(x)}{a_i} > \theta_i} [\Phi(\frac{\Omega(x)}{b_i}) - \Phi(\theta_i)] \right)$$

Therefore, the cumulative loss distribution for the random loading model is given by

$$F_\infty(x) = 1 - \lim_{N\to\infty} p(X \geq x) \quad (6.37)$$
$$= 1 - \left(\Phi[\min(\frac{\Omega(x)}{a_i}, \theta_i)] + 1_{\frac{\Omega(x)}{a_i} > \theta_i} [\Phi(\frac{\Omega(x)}{b_i}) - \Phi(\theta_i)] \right)$$

6.7 Perfect copula

Implied copula approach or so called 'perfect copula' is proposed by Hull and White 2006. This is a 'perfect model ' in terms of its exact fit to the market quotes.

6.7.1 Implied hazard rate paths

Before presenting the specific approach, let us recall the one-factor copula and its implied hazard rate paths. In the chapter 4.2, we derive that

$$P(x_i < x \mid Y = y) = P(\rho_i Y + \sqrt{1-\rho_i^2} \cdot \varepsilon_i < x \mid Y = y)$$
$$= H_i \left[\frac{F_i^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}} \right] \quad (4.4)$$

where H_i is the cumulative distribution function of ε_i.

Therefore, the cumulative probability of the i^{th} default by time t, conditional on the common factor y is

$$P(t_i < t \mid Y = y) = P(x_i < x \mid Y = y) = P(\varepsilon_i < \frac{x - \rho_i Y}{\sqrt{1-\rho_i^2}} \mid Y = y)$$

$$= H_i\left[\frac{F_i^{-1}[Q_i(t)] - \rho_i Y}{\sqrt{1-\rho_i^2}}\right]$$

$$= p_i(y) \quad (4.5)$$

Now, instead of formulating the models in terms of conditional cdf $p_i(y)$, we may use conditional hazard rate. Let $\lambda_i(t \mid y)$ be the hazard rate at time t conditional on common factor Y for obligor i. Then we model the cumulative default probability by assuming that defaults occur according to Poisson process. The relationship between the hazard rate and the cumulative default probability is then

$$p_i(y) = 1 - \exp[-\int_0^t \lambda_i(s \mid y) ds]$$

or equivalently

$$\lambda_i(t \mid y) = \frac{dp_i(y)/dt}{1 - p_i(y)} \quad (6.\ 38)$$

The equation (6. 38) in conjunction with (4.5) can be used to calculate the hazard rate as a function of time t conditional on Y. thus that function is denoted by hazard rate path. The probability that $\lambda_i = \lambda_i(t \mid y)$ is β_i. The calibration problem is to choose the appropriate set of λ_i and its corresponding β_i. In the Hull and White 2006, the details about the specific method for calculating λ_i and β_i as well as some discussions of the hazard rate path on different models are presented.

6.7.2 Implied copula approach

In the previous section in the equation (6. 38), we have seen that the hazard rate paths of obligors are implies by the factor copula model and determined by cdf of common factor Y. Each hazard rate path may represent a future environment so that the set of hazard rate paths may form a distribution of the future environments. Bear in mind that the distribution of the future environments are the key to price the tranche of CDO[8]. Therefore, instead of specifying a copula as we did in the previous chapters, we specify the distribution of the future

[8] the distribution of the future environments are so critical for pricing , since in the valuation process, we have to calculate in a tranche the expected premium leg and default leg for a particular future environment, and then integrated those expected cash flows over the distribution of the future environments.

environments directly. In this sense, it is more advisable to focus on the practical implementation process rather than defining the general models. As introduced in the Hull and White 2006, three general steps are given.

For simplicity, we assume

- All the obligors have the same set of hazard rate path
- The hazard rate is constant along each hazard rate path

Suppose there is a number of different 5-year default rate, and the number denoted by n is sufficiently large[9]. Empirically, it is necessary to include some high default rates and also low default rates. Thus the steps are as follows:

1. Derive the hazard rate path according to each 5 year default rate.
2. Based on the hazard rate path we derive, calculate the present value of premium leg and default leg for each CDO tranche.
3. Select the probabilities β_i to each hazard rate path so that the unconditional expected cash flow of each CDO tranche are zero.

A criticism for the processors is that the solution is not unique, because there are potentially many different distributions of future environments that may fit for the observed market data. In order to avoid any ambiguity by using the approach, Hull and White 2006 has already standardized its implementation in some ways, for those readers who are interested in may look through their paper 'perfect copula'.

So far, besides all those models we reviewed above, there are still several alternative approaches to improve the performance of Gaussian copula model, such as Marshall-Olkin copula in Andersen and Sidenius 2004, random recovery approach in Andersen and Sidenius 2004, etc. With all these models in hand, the problem now arises, that is which of those models the company should eventually select to employ in the real market. The aim of next chapter is to present model comparisons so far based on the performances of abilities of the different models to reproduce market quotes via the DJ iTraxx index. Consequently, one may have some criteria and clues to make judgment on the various models.

[9] Usually, n should be larger than 50.

7. Comparison of various pricing models

In this chapter, some concise model comparisons based on the performances of abilities of the different models to reproduce market quotes via the DJ iTraxx index are presented.

Under the one factor Gaussian copula model, it assumes a flat default correlation structure over the reference portfolio and all the integrals in the pricing formulas can be computed analytically. Mainly due to its obvious advantage of simplicity, this model has become the market standard. However, the existence of the well known correlation smile shows that Gaussian model fails to fit the prices of different CDO tranches simultaneously thus couldn't be an accurate model for CDO valuation. The main explanation of this phenomenon is the lack of tail dependence. Many researchers have proposed different ways to bring more tail dependence into the model and nearly most of them, to some degree, make a better fit to the market data. Such as Student t copula in O'Kane and Schloegl, double t distribution in Hull and White, Marshall-Olkin copula and Gaussian copula with factor loading in Andersen and Sidenius, etc.

According to the Burtschell, Gregory and Laurent 2005, performances of several models (including Gaussian copula model, stochastic correlation extension to Gaussian copula, Student t copula model, double t factor model, Clayton copula and Marshall-Olkin copula) are compared.

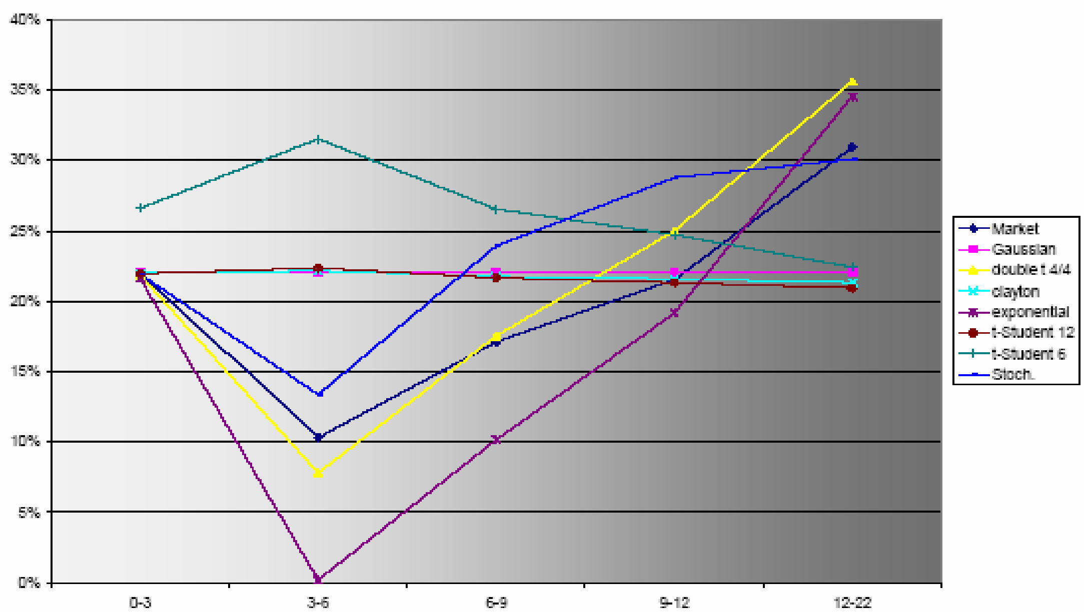
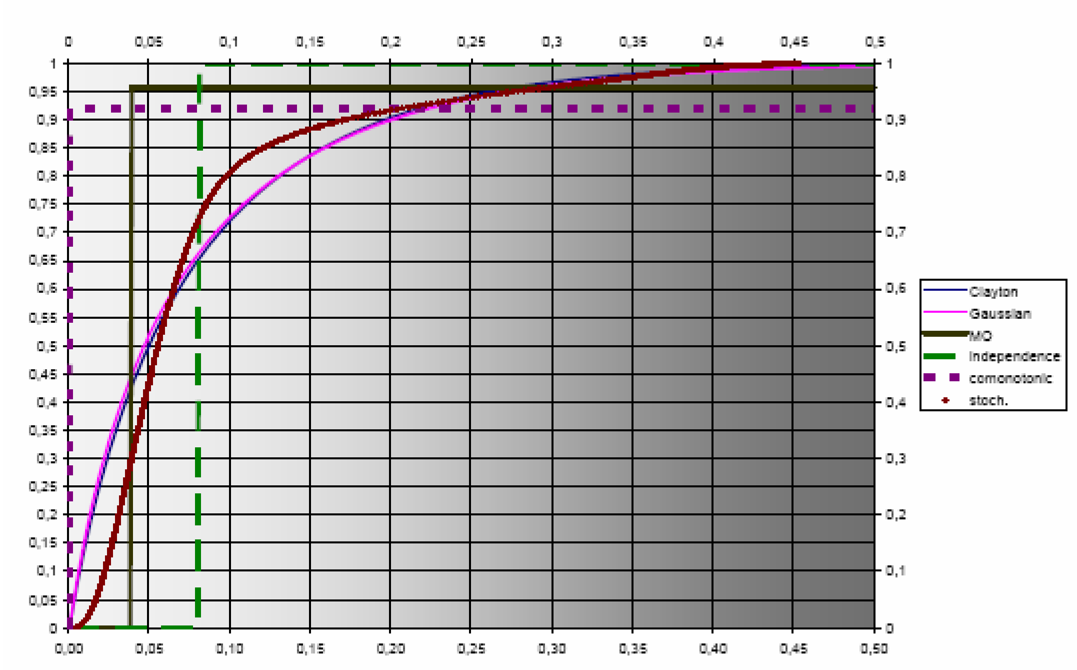


Figure 7.1: implied compound correlations of various models



Graph 7.1: distribution functions of conditional default probabilities
Source: Burtschell, Gregory and Laurent 2005

Comparisons are made starting from the benchmark one factor Gaussian copula. It is obvious to see from the figure 7.1, a flat correlation in the one factor Gaussian copula model associated with a given premium. However, the real market quotes corresponding to each tranche are different and produce a smile skew, which just proof the inaccuracy of the valuation of the standard copula model. In addition, Clayton and Student *t* copulas are both seen to be quite close to Gaussian and thus do not create any correlation smile. They are not precise modeling instruments either. The Marshall-Olkin copula associated with large probabilities of simultaneous jumps leads to strikingly different results and a dramatic fattening of the tail of the loss distributions. Furthermore, the stochastic correlation copula can achieve a reasonable skew, close to that observed in the market (Burtschell, Gregory and Laurent 2005). Last, the double t copula produce a good fit to the market quote.

From Hull and White (2004), it is theoretically and numerically proved that double t distribution copula where both the market factor and the idiosyncratic factor have student t distributions / heavy tails provides a good fit to iTraxx and CDX market data. It is natural to believe that double t copula may possibly be an ideal choice for pricing the tranche of CDOs. However, it has to be noted that it is not possible to compute the distribution function of asset returns analytically because of the lack of stability under convolution. This leads to a dramatic increase of computation time and makes it impossible to use this model for some important applications in practice (Kalemanova et al). Considering this problem, Kalemanova et al has introduced the Normal inverse Gaussian (NIG) copula model. The NIG distribution still has higher tail dependence than the Gaussian distribution. In addition, it is stable under convolution under certain conditions. From the Kalemanova et al, The employment of the NIG distribution does not only bring significant improvement with respect to computation times but also more flexibility in the modeling of the dependence structure of a default structure.

Copula models with random recovery rate (RR) and random factor loadings (RFL) are introduced by Andersen and Sidenius 2004. They both conform better with the observed phenomena (correlation smile) than standard Gaussian model. For the random recovery rate copula, it is shown to produce a heavy upper tail in portfolio loss distributions, however, its effects on generating a skew in tranche values is small and the RR approach is unlikely to become an efficient method. By comparison, factor loading approach did much better in this respect. At reasonable parameter levels, RFL approach is capable of generating correlation skew very similar to the observed data in the market.

Implied copula approach is produced by Hull and White. This approach is slightly different from other copula models (like student t, Clayton, RFL etc) in that they directly specify the distribution of the future environments (hazard rate paths) instead of specifying the copula models as other models do. The outcome of this approach is perfectly good since the model can be exactly fitted to the market quotes for the actively traded CDO tranches. In addition, the model is more intuitive than the base correlation model.

8. Conclusion

This paper gives an overview of the vast areas of credit derivatives and different products in the market, especially the CDO. For reference, I also address the issues about the CDO tranches, loss distribution and large portfolio approximation etc. Basic valuation methodology and the default correlation model framework are discussed in details.

In order to obtain the fair premium of CDO, standard market benchmark model---Gaussian copula model is introduced. Mainly due to the existence of the correlation smile, however, the Gaussian copula model couldn't obtain relatively accurate spreads results. Thus various extension models aiming at improving the performances of fitting the market quotes arise. Nowadays, a good many extension models, such as student t copula, double t copula, stochastic Gaussian copula, Clayton, normal inverse Gaussian copula, RFL Gaussian copula, implied copula model etc , are already introduced in the academic area and nearly all of those models provide a better performances than the industry standard one. Hence companies may select and construct their models based on its own specific requirements and situations. According to Burtschell, Gregory and Laurent 2005, we consider the assessment methodology based on the matching of basket default swap spreads and CDO tranches. For the pricing purpose, once correctly calibrated, student t and Clayton copula models provide similar results as the Gaussian copula models. The Marshall-Olkin copula leads to strikingly different results and a dramatic fattening of the tail of the loss distribution. The double t lies in between and performs well. In addition, it is found out that Copula models with random factor loading (RFL) as well as Normal inverse Gaussian (NIG) copula model can obtain quite similar correlation smile close to market quotes and considered to be two good pricing models. Impressively, the implied copula model has achieved a perfect fit to the real market price and thus becomes a precise model in some situations. However, there are still limitations of those models. For example, for the RFL model, it is argued that the specific parameterizations and model examples used in the paper were rather simplistic, therefore work remains in uncovering parameters and functional forms that best describe the market; for the perfect copula model, it doesn't involve the dynamic evolution of hazard rates or credit spreads and thus not appropriate for some instruments, like valuing an one-year option on a five-year CDO because this depends on hazard rates between years one and five conditional on what we observe happening during the first year. Further researches may be based on improving those problems.

Here it should be kept in mind that the market is at its inception period and all modelling work are inevitably based on a quite limited set of observations, which may not fully representative. Moreover, parameters and models will require revision all the time. Therefore, with the development of the economic market more information about dependence on spread level, maturity, correlation skew etc, will be available. Thus in return, more realistic and hedged performances can be achieved, which leads to further extension and more sophisticated models (Andersen, L. and Sidenius, J. (2004)).

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