VRIJE UNIVERSITEIT AMSTERDAM

MSC BUSINESS ANALYTICS

RESEARCH PAPER

Economic Simulation Engine

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Abstract

The goal of this research is to find the optimal long-term and short-term investment strategy for investing in bonds and stocks. This was done by creating an economic scenario generator application using R shiny. This economic scenario generator simulates stocks and interest rates in an interactive way. For the simulation of stocks, the Heston Stochastic Volatility model was used. For the simulation of interest rates, the two-Additive-Factor Gaussian model was used. Using these models, the optimal weights in bonds and stocks for different investment horizons could be found. The strategy for stocks was buy and hold. For bonds, two strategies were considered. The first was to buy a bond with a maturity equal to the investment horizon. The second strategy was to buy multiple one-year bonds until the end of the investment horizon. Two performance measures were used to investigate the optimal strategy: The Sharpe ratio and the utility function. Three different levels of risk aversion were analyzed when using the utility function. This study found that the optimal short-term investment strategy, using the Sharpe ratio as performance measure, is to invest in bonds. As the horizon increases, more should be invested in stocks depending on bond strategy. Using the utility function as performance measure, the optimal short-term and long-term strategy is to invest in stocks if the investor is a risk seeker or risk neutral. When the investor is risk averse, the optimal short-term strategy is to invest in stocks and as the horizon increases, more should be invested in bonds depending on bond strategy. The study shows that if the multiple one-year bond strategy is used, a higher percentage should be invested in bonds compared to the one H-year bond strategy.

Contents

1. Introduction

Simulating economic scenarios are becoming more and more important for a lot of financial institutions (e.g. banks, insurance companies) (Charpentier, 2014). Using the simulations of e.g. stock price and interest rates, financial institutions can invest in an optimal way to maximize their profits. Also, for economists it can be useful to study interest rates to predict the economy. For example, they may be able to predict, with a certain probability, if there possibly will be an economic recession. So simulated scenarios will give an indication about future stock prices and interest rates.

Stochastic differential equations are used to simulate stock prices and interest rates. A stochastic differential equation is different from an ordinary differential equation in the sense that it contains a stochastic element. So in these equations, at least one term represents a random process (e.g. Wiener process for Gaussian random variable). This is useful for simulating e.g. stock prices since financial analysts assume that the movement of stock prices involve a random component.

There exist a lot of models for modeling economic scenarios. An example of a stochastic differential equation that models stock price is *Geometric Brownian Motion* (Black and Scholes, 1973). This is one of the most frequently used models for stock price modeling. The stochastic process of the stock price $S(t)$ is given by the dynamics

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{1.1}$$

Where $W(t)$ is a Wienier process, μ is the drift and σ is volatility. μ and σ are called the **parameters** or **variables** of the model. In Figure 1.1 two realizations with different parameters of the Geometric Brownian Motion process are shown.

It can be observed that these parameters play an important role in the simulation. When the volatility (σ) is greater, there are more fluctuations in stock price. The greater the drift (μ), the higher the stock return.
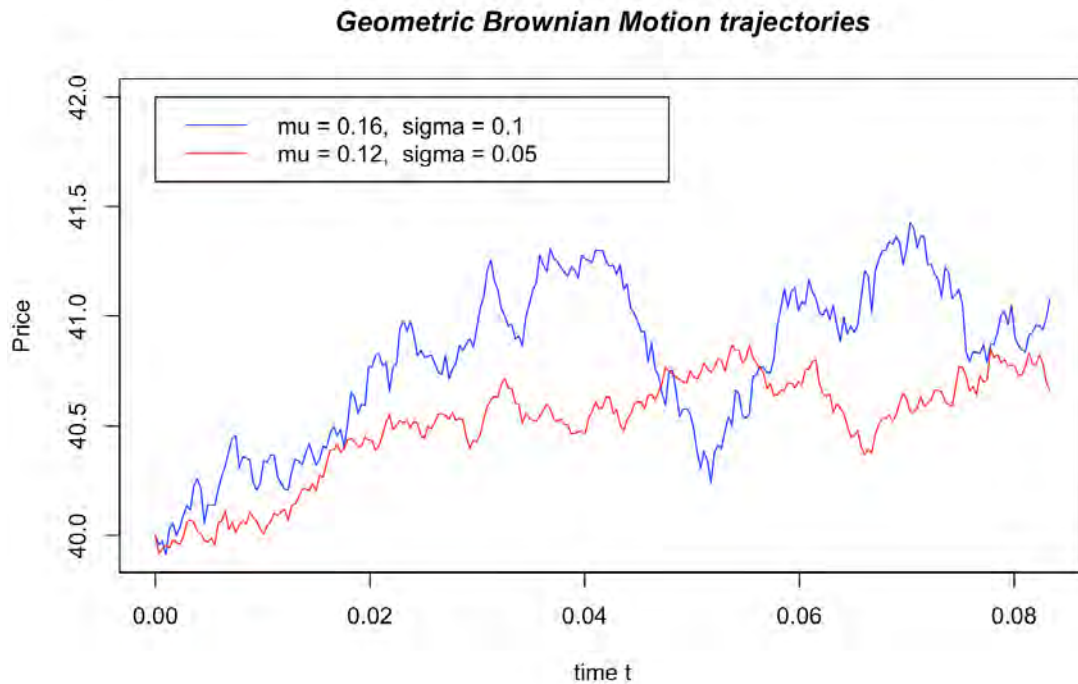
FIGURE 1.1: Realizations of the GBM process. S(0) = 40.

The goal of the research paper is to investigate the optimal investment strategy in a simple portfolio consisting of bonds and stocks given different economic scenarios for these products. These economic scenarios are generated by simulating stock prices with *Heston Stochastic Volatility model* and interest rates with the *two-factor additive Gaussian G2++ short rate model*. By analyzing portfolio return with different weights in bonds and stocks, the optimal investment strategy can be found for different investment horizons. For stocks a *buy and hold* strategy is applied. For bonds two strategies are considered. The first was to buy a bond with a maturity equal to the investment horizon. The second strategy was to buy multiple one-year bonds until the end of the investment horizon. The optimal investment strategy is considered the strategy with the highest *utility* and *Sharpe ratio*. The research question of the paper is:

What is the optimal long-term and short-term investment strategy for investing in bonds and stocks?

For the research paper, an interactive web application will be made to simulate economic scenarios. This tool will be developed using *R Shiny*. R shiny is an *R* package which allows people to build interactive applications. The application that will be developed allows users to visualize the simulations and portfolio results in an interactive way. The user can interact with the application by changing the model parameters and simulations parameters and providing investment decisions.

Multiple studies have been conducted to investigate portfolio performance using stochastic differential equations (e.g. Geometric Brownian Motion) for simulation of stock price and/or interest rates (Reddy and Clinton, 2016). Instead of using these classical models,

this paper will use more sophisticated/realistic models for the simulation engine. Grzelak, Oosterlee, and Van Weeren (2011) also combine the Heston model to simulate stock prices and the Gaussian short rate model to simulate interest rates. They used this for pricing derivatives. This paper however, will focus on simulating economic scenarios and combining this with an interactive R shiny tool.

Many studies also investigated the dependence of the investment horizon on asset allocation. Dierkes, Erner, and Zeisberger (2010) found that investing in bonds is preferred for a short horizon, but stocks show an outperformance for longer horizons. Also Veld-Merkoulova (2011) found that a longer investment horizon leads to an increasing percentage of investing in more risky assets (i.e. stocks).

The rest of the paper is structured as follows. The next chapter is about the simulation of stocks. The following chapter is about the simulation of interest rates. Then follows a chapter about portfolio theory and investment strategies. In chapter 5, the parameters of models are discussed. Chapter 6 contains the results of the research. Then follows the discussion and the conclusion. The paper ends with the instruction manual for the investment tool.

2. Simulation of Stocks

This chapter is about the simulation of stocks. It starts with the definition of stocks followed by the reasoning behind the chosen model. Next, the theoretical model is discussed in detail. Then the actual simulation of stocks is explained.

2.1 Definition of Stock

A stock represents a portion of ownership of a company. A stock holder of a company is entitled to a part of the assets and earnings of that company.

2.2 Heston Stochastic Volatility Model

There are many models for simulating stock prices. An example of a model is the Black-Scholes model which was mentioned in Section 1 (Equation 1.1). But for this paper the Heston Stochastic Volatility Model is used. It is similar to the geometric brownian motion. However, it is a stochastic volatility model which means that it allows for time-varying volatility by saying that the volatility is driven by its own stochastic process (Heston, 1993). It thereby tackles one of the biggest limitations of the Black-Scholes model which assumes that the variance is constant (Black and Scholes, 1973). So the reason why the model is popular is because of the attention to the correlation between the process of the stock price and the process of the volatility. The Heston model was not the first stochastic volatility model but it is one of the most important and widely used model for pricing options (Rouah, 2013; Mrázek and Pospíšil, 2017).

2.3 Theoretical Heston Model

Under the Heston Stochastic Volatility Model, the stock price is described as

$$dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)dW_1(t) \tag{2.1}$$

And the variance is described as

$$dV(t) = k(\theta - V(t))dt + \eta\sqrt{V(t)}dW_2(t) \tag{2.2}$$

Where

$$dW_1(t)dW_2(t) = \rho dt \tag{2.3}$$

Where $\rho \in [-1, 1]$.

The variables are defined as:

- **S(t)**: Stock price at time t

- μ: Drift

- **t**: Time

- *dt*: Indefinitely small increment of time

- **V(t)**: Variance of the stock price

- η: Volatility of the volatility process of the stock price

- θ: Long-term price variance

- *k*: Reversion to the long-term price variance

- $W_1(t)$: Wiener process of the stock price component

- $W_2(t)$: Wiener process of the stock price volatility component.

- ρ: Correlation between the Wiener processes

The Heston model is similar to the Black-Scholes model since the stock price follows a similar Black-Scholes process (2.1). However, it can be observed that the variance is modeled as a square-root process that is correlated with the stock price and it follows a Cox-Ingersoll-Ross (Cox, Ingersoll Jr, and Ross, 1985) process (2.2).

2.4 Simulating the Heston Model

In the Heston model, the stock price and the variance both follow a continuous process. So to be able to simulate this, the Heston model should be *discretized*. Discretization is the process of transferring a continuous model to a discrete model. So this is done to approximate a numerical solution for a stochastic differential equation. A common method for discretization is the *Euler Discretization*.

2.4.1 Euler Discretization

A solution to the Euler discretization for the stock of the Heston model is

$$S(t_{i+1}) = S(t_i) - \mu S(t_i)\Delta t + \sqrt{V(t_i)}S(t_i)\sqrt{\Delta t}Z_1(t_i) \tag{2.4}$$

And for the variance is

$$V(t_{i+1}) = V(t_i) - k(V(t_i) - \theta)\Delta t + \eta\sqrt{V(t_i)}\sqrt{\Delta t}Z_2(t_i) \tag{2.5}$$

Where

$$Z(t_i) = \begin{bmatrix} Z_1(t_i) \\ Z_2(t_i) \end{bmatrix}, \qquad Z \sim N(0, \Sigma), \qquad \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{2.6}$$

And where Δt is the simulation frequency (e.g. yearly, monthly, daily).

However, as Gatheral (2011) pointed out, because $Z_2(t_i) \sim N(0, 1)$, the variance process (Equation 2.5) will often be negative which is not desired. A solution to this problem is to use *Milstein discretization*. Haugh (2004) also notes that the Euler scheme, in combination with the Heston model, can perform poorly in practice .

2.4.2 Milstein Discretization

According to Gatheral (2011), the Milstein discretization is the most preferred discretization scheme to simulate the Heston process. Using this scheme, the stock process, by implementing Itô-Taylor expansion, can be described as

$$S(t_{i+1}) = S(t_i) * \exp\left(\left(\mu - \frac{V(t_i)}{2}\right)\Delta t + \sqrt{V(t_i)\Delta t}Z_1(t_i)\right) \tag{2.7}$$

And the variance process can be described as

$$V(t_{i+1}) = \left(\sqrt{V(t_i)} + \frac{\eta}{2}\sqrt{\Delta t}Z_2(t_i)\right)^2 + k(\theta - V(t_i))\Delta t - \frac{\eta^2}{4}\Delta t \tag{2.8}$$

Where

$$Z(t_i) = \begin{bmatrix} Z_1(t_i) \\ Z_2(t_i) \end{bmatrix}, \qquad Z \sim N(0, \Sigma), \qquad \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{2.9}$$

When we compare Equation 2.5 and 2.8, the occurrences of negative variances are are substantially reduced in the case of using Milstein discretization. As one can see in Equation 2.8, this is caused by the square.

2.5 Output of the Simulation

The simulation will result in the following matrix for the stock price process

$$S = \begin{bmatrix} S(t_1)^1 & \cdots & S(t_1)^j & \cdots & S(t_1)^N \\ \vdots & \ddots & & & \vdots \\ S(t_i)^1 & & S(t_i)^j & & \\ \vdots & & & \ddots & \vdots \\ S(t_M)^1 & \cdots\cdots\cdots\cdots\cdots & & & S(t_M)^N \end{bmatrix} \qquad \begin{aligned} i &= 1, \ldots, M \\ j &= 1, \ldots, N \end{aligned} \tag{2.10}$$

Where N is the amount of simulation paths and M is the time horizon for each simulation path.

A similar matrix V is constructed for the process of the variance. This matrix has the same dimensions (N x M) as matrix S.

3. Simulation of Interest Rate

This chapter is about the simulation of interest rate. It starts with some fundamental theory of interest rate. Then the reasoning behind the chosen model is discussed. Next, the theoretical model that is used for simulation will be explained. Afterwards, the simulation of interest rate is discussed. The chapter ends with theory on the pricing of zero-coupon bonds and extracting the market yield curve.

3.1 Fundamental Theory of Interest Rate

Interest rate is defined as *"the proportion of a loan that is charged as interest to the borrower, typically expressed as an annual percentage of the loan outstanding"*(Dictionary, 2018). Closely following Brigo and Mercurio (2007), the first definition that is discussed concerning interest rate is the **bank account**. Assume $B(t)$ is the value of the money in the bank at time t, then the process of $B(t)$ can be described by the stochastic differential equation

$$dB(t) = r(t)B(t), \quad B(0) = 1 \tag{3.1}$$

It follows that

$$B(t) = \exp\left(\int_0^t r(s)ds\right) \tag{3.2}$$

As can be observed in equation 3.2, the value of money accrues with **short rate** $r(t)$. The difference in between the value of the bank account between two different time instances t and T is then described as

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r(s)ds\right) \tag{3.3}$$

$D(t,T)$ is called the **discount factor**.

The next concept that is discussed is a **bond**. A bond is a contract where the borrower has to pay the bondholder coupon payments on certain dates and has to pay face value when the contract ends. The duration of the contract is called the **maturity of the bond**. A **zero-coupon bond**, as the name suggests, is a bond with a certain maturity T that pays no coupons. The **price of a zero-coupon bond** at time t is denoted as $P(t,T)$. The price of a zero-coupon bond and the discount factor are closely linked. When rates are deterministic, $P(t,T) = D(t,T)$. When rates are stochastic, $P(t,T) = \mathbb{E}_t^M[D(t,T)]$ given probability measure M.

Furthermore, the **continuously-compounded zero-coupon interest rate** at time t for maturity T is described as

$$R(t,T) = -\frac{\ln P(t,T)}{\tau(t,T)} \tag{3.4}$$

Where $\tau(t, T)$ is the time to maturity $(T - t)$. $R(t, T)$ is a rate which accrues *continuously*. There are other ways to construct the zero-coupon interest rate e.g. annually-compounded where the rate accrues annually. With $R(t, T)$ a **zero-coupon yield curve** can be constructed. This curve shows the relationship between bond yields and time to maturity. The yield curve is also called the term structure of interest rates.

The last definition that needs to be discussed is the **instantaneous forward interest rate**. At time t with maturity T this is defined as

$$f(t, T) = \lim_{S \to T^+} F(t; T, S) = -\frac{\delta \ln P(t, T)}{\delta T} \tag{3.5}$$

This can intuitively be explained as a simply-compounded forward rate $F(t; T, S)$ where expiry T and maturity S are extremely close to each other.

3.2 Two-Additive-Factor Gaussian Model G2++

For the simulation of interest rates, the Two-Additive-Factor Gaussian Model G2++ (short rate) model is used. This model describes the future evolution of the short rate. Brigo and Mercurio (2007) showed that this model is similar to the two-factor Hull-White model, but more easily implementable.

The reason a *two*-factor model is used, is because two-factor models are in general preferred over one-factor models since they create more realistic interest rate simulations (Deventer, 2011). When one-factor models are used, interest rates for all maturities in the yield curve are perfectly correlated since interest rates depend on one stochastic term. However, in reality interest rates are not perfectly correlated. Another reason why the G2++ model is used for simulation is because it is very useful in practice (Brigo and Mercurio, 2007). Furthermore, the model has efficient numerical procedures for pricing any fixed-income security.

3.3 Theoretical G2++ Short Rate Model

In the G2++ model the short rate is defined as

$$r(t) = x(t) + y(t) + \phi(t) \tag{3.6}$$

Where

$$dx(t) = -ax(t)dt + \sigma dW_x(t), \qquad x(0) = 0 \tag{3.7}$$

$$dy(t) = -bx(t)dt + \eta dW_y(t), \qquad y(0) = 0 \tag{3.8}$$

Where $a, b, \sigma, \eta > 0$ and

$$dW_x(t)dW_y(t) = \rho dt \quad \text{with} \quad \rho \in [-1, 1] \tag{3.9}$$

And where

$$\phi(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 + \frac{\eta^2}{2b^2}(1 - e^{-bt})^2 + \rho\frac{\sigma\eta}{ab}(1 - e^{-at})(1 - e^{-bt}) \quad (3.10)$$

Where the market-observed instantaneous forward rate is given by

$$f^M(0,t) = -\frac{\delta \ln P^M(0,t)}{\delta t} \quad (3.11)$$

The variables are defined as:

- a: Mean reversion speed for the first factor

- b: Mean reversion speed for the second factor

- dt: Indefinitely small increment of time

- σ: Volatility of the first factor

- η: Volatility of the second factor

- W_x: Wiener process of the first factor

- W_y: Wiener process of the second factor

- ρ: Correlation between the Wiener processes

- $P^M(0,t)$: Market-observed term structure of discount factors

Interest rates are characterized by a *mean reversion* which means that interest rates revert to a long-term equilibrium level as time goes by. These parameters (*a* and *b*) are also called the drift. So the G2++ model takes this into account. Furthermore, as can be observed in Equation (3.6), in the G2++ model the process of the short rate is driven by of two correlated Gaussian Ornstein-Uhlenbeck (Doob, 1942) (3.7 and 3.8) factors plus a deterministic shift function (3.10).

3.4 Simulating the G2++ Model

Similar to the simulation of stocks, the first and second factor are both a continuous process. So to simulate interest rates, the G2++ model should also be discretized. Also for the G2++ model, the *Euler discretization* is a common method (Aziezeh, 2017).

3.4.1 Euler Discretization

Using the Euler discretization, the short rate process be described as

$$r(t_{i+1}) = x(t_{i+1}) + y(t_{i+1}) + \phi(t_{i+1}) \quad (3.12)$$

Where

$$x(t_{i+1}) = x(t_i) - ax(t_i)(t_{i+1} - t_i) + \sigma\sqrt{(t_{i+1} - t_i)}Z_x(t_i) \tag{3.13}$$

$$y(t_{i+1}) = y(t_i) - by(t_i)(t_{i+1} - t_i) + \eta\sqrt{(t_{i+1} - t_i)}Z_y(t_i) \tag{3.14}$$

$$\phi(t_{i+1}) = f^M(0, t_{i+1}) + \frac{\sigma^2}{2a^2}(1 - e^{-at_{i+1}})^2 + \frac{\eta^2}{2b^2}(1 - e^{-bt_{i+1}})^2 + \\ \rho\frac{\sigma\eta}{ab}(1 - e^{-at_{i+1}})(1 - e^{-bt_{i+1}}) \tag{3.15}$$

Where

$$Z(t_i) = \begin{bmatrix} Z_x(t_i) \\ Z_y(t_i) \end{bmatrix}, \qquad Z \sim N(0, \Sigma), \qquad \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{3.16}$$

However, in addition to the monte carlo error, this will also lead to a discretization error (Aziezeh, 2017). This is why an *exact solution for Ornstein-Uhlenbeck* is used instead of the Euler discretization.

3.4.2 Exact solution for Ornstein-Uhlenbeck

The derivation of the exact solution of the SDE for an Ornstein-Uhlenbeck process can be found in the appendix (B.1). This derivation is applied to Equations 3.7 and 3.8. The short rate process can then be described as

$$r(t_{i+1}) = x(t_{i+1}) + y(t_{i+1}) + \phi(t_{i+1}) \tag{3.17}$$

Where

$$x(t_{i+1}) = x(t_i)e^{-a(t_{i+1} - t_i)} + \sigma\sqrt{\frac{1 - e^{-2a(t_{i+1} - t_i)}}{2a}}Z_x(t_i) \tag{3.18}$$

$$y(t_{i+1}) = y(t_i)e^{-b(t_{i+1} - t_i)} + \eta\sqrt{\frac{1 - e^{-2b(t_{i+1} - t_i)}}{2b}}Z_y(t_i) \tag{3.19}$$

$$\phi(t_{i+1}) = f^M(0, t_{i+1}) + \frac{\sigma^2}{2a^2}(1 - e^{-at_{i+1}})^2 + \frac{\eta^2}{2b^2}(1 - e^{-bt_{i+1}})^2 + \\ \rho\frac{\sigma\eta}{ab}(1 - e^{-at_{i+1}})(1 - e^{-bt_{i+1}}) \tag{3.20}$$

Where

$$Z(t_i) = \begin{bmatrix} Z_x(t_i) \\ Z_y(t_i) \end{bmatrix}, \qquad Z \sim N(0, \Sigma), \qquad \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{3.21}$$

Using this method, there is no discretization error (Aziezeh, 2017).

3.5 Output of the Simulation

The simulation will result in two (N x M) matrices X and Y for the process of the first and second factor. N is the amount of simulation paths and M is the time horizon for each simulation path. These matrices are similar to the matrix shown in Equation 2.10.

3.6 The Pricing of Zero-Coupon bond

Following Brigo and Mercurio (2007), the zero-coupon bond price at time t with a maturity T for the two-factor additive Gaussian model is given by

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} e^{A(t,T)} \tag{3.22}$$

Where

$$A(t,T) = \frac{1}{2} \left[V(t,T) - V(0,T) + V(0,t) \right] \\ - \frac{1 - e^{-a(T-t)}}{a} x(t) - \frac{1 - e^{-b(T-t)}}{b} y(t) \tag{3.23}$$

$$V(t,T) = \frac{\sigma^2}{a^2} \left[T - t + \frac{2}{a} e^{-a(T-t)} + \frac{1}{2a} e^{-2a(T-t)} - \frac{3}{2a} \right] \\ + \frac{\eta^2}{b^2} \left[T - t + \frac{2}{b} e^{-b(T-t)} + \frac{1}{2b} e^{-2b(T-t)} - \frac{3}{2b} \right] \\ + 2\rho \frac{\sigma\eta}{ab} \left[T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b} - \frac{e^{-(a+b)(T-t)} - 1}{a+b} \right] \tag{3.24}$$

Where $x(t)$ and $y(t)$ are the two factors used in the G2++ model. $V(t,T)$ can be interpreted as the volatility of $P(t,T)$ where the first term represent the variance of the first factor, the second term the variance of the second factor and the third term as the covariance between the two factors. $P(t,T)$ will result in a (N x M x T) matrix where N is the amount of simulation paths, M is the time horizon for each simulation path and T is the corresponding maturity.

3.7 The Market Term Structure

As mentioned before, for pricing a zero-coupon bond, it is needed to construct a yield curve. As can be observed in Equation 3.22, the market-observed discount factor $P^m(0,T)$ is required. Furthermore, to simulate interest rates using the G2++ short rate model, the market-observed instantaneous forward rate $f^m(0,T)$ is required (3.10). This section explains how these two components are obtained.

3.7.1 Obtaining market-observed discount factor $P^m(0, T)$

The market-observed discount factor is defined as

$$P^m(0, T) = e^{-R^m(0,T)\tau(0,T)} \tag{3.25}$$

Where $\tau(t, T)$ is time to maturity ($= T - t$). As can be observed in Equation 3.25, the market zero rates ($R^m(0, T)$) are needed to calculate $P^m(0, T)$. With those rates, a cubic spline function can be made which can be used to make interpolation continuous and therefore more dynamical. With the function, the interpolated continuously compounded zero-coupon rates can be extracted. This leads to

$$R^m(0, T) \approx R^m_{\text{spline}}(0, T) = a_1(T - t_1)^3 + a_2(T - t_1)^2 + a_3(T - t_1) + a_4 \tag{3.26}$$

Where a_1, a_2, a_3, a_4 are the cubic spline coefficients, t_1 is the lower bound of the corresponding time-interval. The cubic spline coefficients are obtained by using the R function *splinefun* on the market zero rates. By combining everything, the market-observed discount factor is given by

$$P^m(0, T) = e^{-T\left[a_1(T-t_1)^3 + a_2(T-t_1)^2 + a_3(T-t_1) + a_4\right]} \tag{3.27}$$

3.7.2 Obtaining market-observed instantaneous forward rate $f^M(0, T)$

The market-observed instantaneous forward rate is given by

$$
\begin{aligned}
f^m(0, T) &= \frac{\delta \ln P^m(0, T))}{\delta T} = -\frac{\delta \left[-R^m(0, T)\tau(0, T)\right]}{\delta T} \\
&= \frac{\delta \left[-R^m(0, T)\tau(0, T)\right]}{\delta T} = R^m(0, T) + \tau(0, T)\frac{\delta R^m(0, T)}{\delta T}
\end{aligned}
\tag{3.28}
$$

Where $\tau(0, T)$ is time to maturity ($= T$). As can be observed in Equation 3.28, again the market zero rates ($R^m(0, T)$) are needed to calculate $f^m(0, T)$. This is done in Equation 3.26. The partial derivative of this equation with respect to T of Equation 3.26 is given by

$$\frac{\delta R^m(0, T)}{\delta T} \approx \frac{\delta R^m_{\text{spline}}(0, T)}{\delta T} = 3a_1(T - t_1)^2 + 2a_2(T - t_1)^2 + a_3 \tag{3.29}$$

By combining everything, the market-observed instantaneous forward rate can be written as

$$
\begin{aligned}
f^m(0, T) &= a_1(T - t_1)^3 + a_2(T - t_1)^2 + a_3(T - t_1) + a_4 \\
&\quad + T\left[3a_1(T - t_1)^2 + 2a_2(T - t_1)^2 + a_3\right]
\end{aligned}
\tag{3.30}
$$

4. Portfolio Theory & Strategies

4.1 From Simulation to Portfolio Return

An important step to answer the research question is to be able to calculate stock returns and zero-coupon interest rates from the simulations. Then it is possible to calculate the total return of the portfolio. The return on stocks of a simulation i with a horizon H is calculated by

$$r_s^i = \sum_{j=2}^{H} \log \left(\frac{S_j^i}{S_{j-1}^i} \right) \qquad i = 1, \ldots, N \tag{4.1}$$

The zero-coupon interest rate at time t with maturity T of simulation i with horizon H is calculated (using Equation 3.4) the following way

$$z^i = \begin{cases} \prod_{\substack{0 \leq t \leq H-T \\ t \text{ is a multiple of T}}} \left(\left(1 + R^i(t, t+T)\right)^T \right) - 1, & \text{if } H\%T = 0 \\[2em] \begin{cases} \left(1 + R^i(H-1, (H-1)+T)\right) - 1 & \text{if } H < T \\[1em] \left(1 + R^i(H-1, (H-1)+T)\right) \left(\prod_{\substack{0 \leq t \leq H-T-1 \\ t \text{ is a multiple of T}}} \left(\left(1 + R^i(t, t+T)\right)^T \right) \right) - 1 & \text{if } H > T \end{cases} & \text{if } H\%T > 0 \end{cases} \tag{4.2}$$

Where '$H\%T$' means the *remainder* of H/T. If for example the maturity T is 2 years and H is 7 years, the zero-coupon interest rate is given by

$$z^i = \left(1 + R^i(0,2)\right)^2 * \left(1 + R^i(2,4)\right)^2 * \left(1 + R^i(4,6)\right)^2 * \left(1 + R^i(6,8)\right) - 1 \tag{4.3}$$

The total return of simulation i is calculated by multiplying the weight in bonds times the zero-coupon interest rate plus the weight in stocks times the stock return:

$$r_p^i = w_b * z^i + w_s * r_s^i \qquad i = 1, \ldots, N \tag{4.4}$$

Where r_p^i is the return of the portfolio, w_b is the weight in bonds and w_s is the weight in stocks ($= 1 - w_b$). Using the vector r_p, a lot of statistics can be calculated. These statistics include the standard deviation of the returns, the expected return, the Sharpe ratio (see Section 4.2) and many more.

4.2 Portfolio Performance Measures

To find the optimal long-term and short-term investment strategy for investing in bonds and stocks, there needs to be considered what is 'optimal'. This is done by choosing an

appropriate performance measure. One of the most widely used measure for portfolio performance is the *Sharpe ratio* (Sharpe, 1966). This performance measure S is given by

$$S = \frac{\mathbb{E}[r_p] - r_f}{\sigma_p} \tag{4.5}$$

Where $\mathbb{E}[r_p]$ is the expected portfolio return, r_f is the risk-free rate and σ_p is the standard deviation of the portfolio. As can be seen in Equation 4.5, the higher the Sharpe ratio, the higher the portfolio returns ($\mathbb{E}[r_p] - r_f$) have been relative to the risk (σ_p). However, Levy (2017) pointed out that the Sharpe ratio is not preferable since the it assumes that unlimited borrowing at the risk- free rate is possible. He suggested to use the *geometric mean* which is given by

$$GM = \left(\prod_{i=1}^{N} a_i \right)^{\frac{1}{N}} \qquad i = 1, \ldots, N \tag{4.6}$$

Where a_i is the portfolio return of simulation i and N is the amount of simulations. The geometric mean can be approximated as

$$GM_{\text{approx}} \approx \mu_p - \frac{\sigma_p^2}{2} \tag{4.7}$$

Where μ_p is the arithmetic mean (expected return) of the portfolio and σ^2 is the variance of the portfolio. According to Levy (2017), the geometric mean is superior to the Sharpe ratio in realistic settings. However, as can be observed in Equation 4.7, the approximation of the geometric mean is in fact the **utility** function with a *risk aversion* of 1 since utility is given by

$$U = \mathbb{E}[r_p] - \frac{\sigma_p^2 A}{2} \tag{4.8}$$

Where $\mathbb{E}[r_p]$ ($= \mu_p$ in Equation 4.7) is the expected return of the portfolio and A is the investor's degree of risk aversion. The larger the value of A, the more risk averse the investor, since the variance (risk) will be more dominant in Equation 4.8. Thus, this is a called a *risk-averse* investor (Bodie, Kane, and Marcus, 2012). A *risk-neutral* investor ($A = 0$) will only look at the expected return without caring about the risk. A *risk seeker* is an investor where $A < 0$.

Both the Sharpe ratio and the utility function will be used as a portfolio performance measure. So the highest utility or Sharpe ratio of a portfolio is considered to be the optimal portfolio. Three different levels of risk aversion for the analysis of utility will be used: $A = -10$, $A = 0$ and $A = 10$.

4.3 Investment Horizon

To answer the research question, the optimal investment strategy for different horizons must be determined. Therefore, the Sharpe ratio and the utility function (and other statistics) will be calculated for every year up to 30 years in the future.

4.4 Stock strategy

The investment strategy that will be used for the paper concerning stocks is a **buy and hold** strategy. This means that stocks will be bought and hold for the entire investment horizon and will then sold. This strategy is used for simplicity.

In this research, the stocks will be invested in the *MSCI World Index* which represents stock performance of large and mid-cap of 23 developed markets countries[1]. A reason for investing in the MSCI world index is simplicity. When investing in separate stocks, a covariance matrix must be constructed for the simulation engine. When investing in the MSCI world index, the covariance between the stocks is already incorporated implicitly. Therefore, the efficient market hypothesis is followed (Bodie, Kane, and Marcus, 2012). This hypothesis states that the market can not be outperformed.

4.5 Bond strategy

For bonds there are two strategies. The first one, similar to stocks, is the buy and hold strategy. This means that an investor buys a bond that has the same maturity as the investment horizon. The second strategy is to buy a bond with a maturity of one year every year until the end of the investment horizon. An example of the two different strategies with an investment horizon of four years is shown in Figure 4.1. With this two different strategies there can be investigated whether the **maturity risk premium** is worth an investors while. This premium is a compensation for the investment with the longer bond maturity (strategy 1). This is because the investor has to tolerate extra risk for owning a bond with a longer maturity.
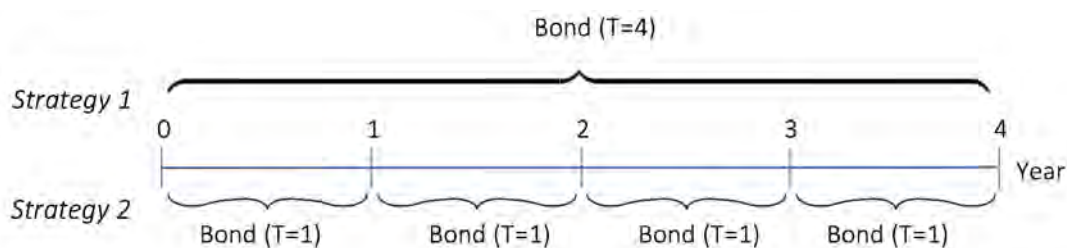

FIGURE 4.1: An example of the two different bond strategies. The investment horizon is four years.

In this research, there will be invested in theoretical zero coupon bonds based on the ATM Euro cap interest rate curve.

4.6 Number of Simulations

The amount of simulations is 500. This amount is enough to obtain accurate results while having an acceptable computation time.

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[1] Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Hong Kong, Ireland, Israel, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Singapore, Spain, Sweden, Switzerland, the UK and the US.

5. Model Parameters

As mentioned before, the model parameters play an important role in the simulation for stock prices and interest rates. The Heston Stochastic Volatility model and the two-factor additive Gaussian G2++ short rate model both have their own model parameters which need to be estimated. To get accurate results, appropriate values for these parameters must be estimated. To obtain the right values, the parameters can be calibrated to market data. This section will discuss the calibration procedure of the model parameters in more detail.

5.1 $\mathbb{P}-$ and $\mathbb{Q}-$Measure

When simulating, the probability distribution of the market prices in the future needs to be estimated. This modeling of the real probability distribution of the market prices is called simulating under $\mathbb{P}$. When the probability is measured in such a way that todays value of an asset is the sum of the expected future payoffs discounted at the risk free rate, the $\mathbb{Q}$-measure is used (Meucci, 2011). This measure assumes that the market is free of arbitrage. Thus, the $\mathbb{P}$-measure is preferred for an economic scenario generator and the $\mathbb{Q}$-measure is preferred for pricing derivatives.

5.2 Heston Parameters

The calibration of the Heston model on the MSCI world index was done in a previous masters thesis by Kooistra (2017). Kooistra calibrated the parameters on monthly data from 2005 until the end of 2016. Thus, the resulting parameters can be used for this paper, since they are based on recent data and are calibrated using the $\mathbb{P}$-measure. Kooistra (2017) used the Method of Simulated Moments to obtain a simulation distribution that acts as the empirical distribution of MSCI world index. The parameters that are used can be observed in Table 5.1.

TABLE 5.1: Parameters for the Heston model.

Parameter	Value
μ	0.0627
θ	0.0206
η	0.1206
V_0	0.0046
ρ	0.9949
k	0.3707

5.3   G2++ Parameters

The two-factor additive Gaussian G2++ short rate model was previously calibrated by Moudiki (2014). Moudiki (2014) calibrated the model to ATM Euro caps prices observed on December 31, 2011. This means the parameters were calibrated under the Q-measure, which is not preferred. Unfortunately, no studies could be found where the G2++ model was calibrated under the $\mathbb{P}$-measure using recent data. The parameters that are used for the G2++ models can be observed in Table 5.2.

TABLE 5.2: Parameters for the two-factor additive Gaussian G2++ short rate model.

Parameter	Value
a	0.5000
b	0.3541
σ	0.0942
η	0.0843
ρ	-0.9986

6. Results

This chapter contains the results of the research. The results for both performance (Sharpe ratio and utility) measures are shown.

6.1 Sharpe Ratio

In Figure 6.1 the optimal weight in bonds, using the Sharpe ratio, against the investment horizon is shown. When the one H-year bond strategy (strategy 1) is used, it can be observed that if the investment horizon is one year, 100% should be invested in bonds. When the horizon increases, more should be invested in stocks. When the horizon is eight years or more, 100% should be invested in stocks.

Also in Figure 6.1, the multiple one-year bond strategy (strategy 2) is shown. Similar to the other strategy, 100% should be invested in bonds if the horizon is one year and this percentage goes down as the horizon increases. However, when the horizon is eight years, the optimal percentage in bonds remains around 40 percent.
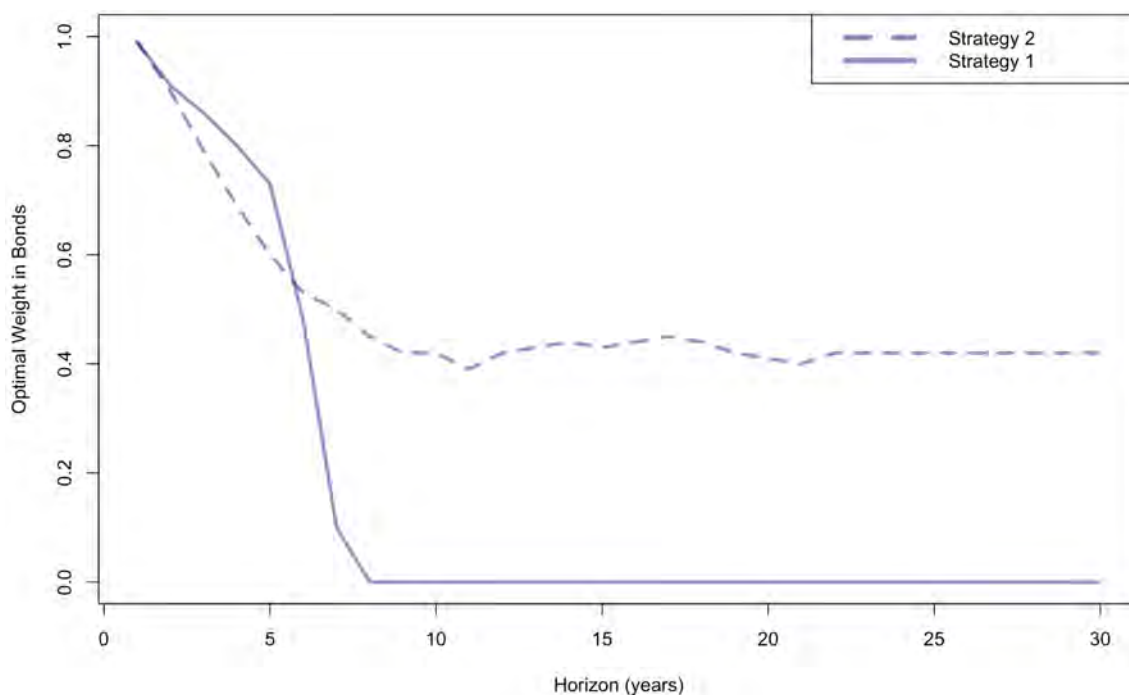

FIGURE 6.1: Optimal weight in bonds (according to the Sharpe ratio) against the investment horizon for both strategies. Strategy 1: One H-year bond. Strategy 2: Multiple (H) one-year bonds.

6.2 Utility Function

In Figure 6.2 the optimal weight in bonds for the three different levels of risk aversion, using utility, against the investment horizon is shown. It can be observed that when an investor is a risk seeker ($A = -10$) or risk neutral ($A = 0$), the investor should never invest in bonds. When using the one H-year bond strategy and if an investor is risk averse (A=10), the investor should only invest in stocks if the investment horizon is 14 years or less. However, if the horizon increases, more percent of the portfolio should be invested in bonds.
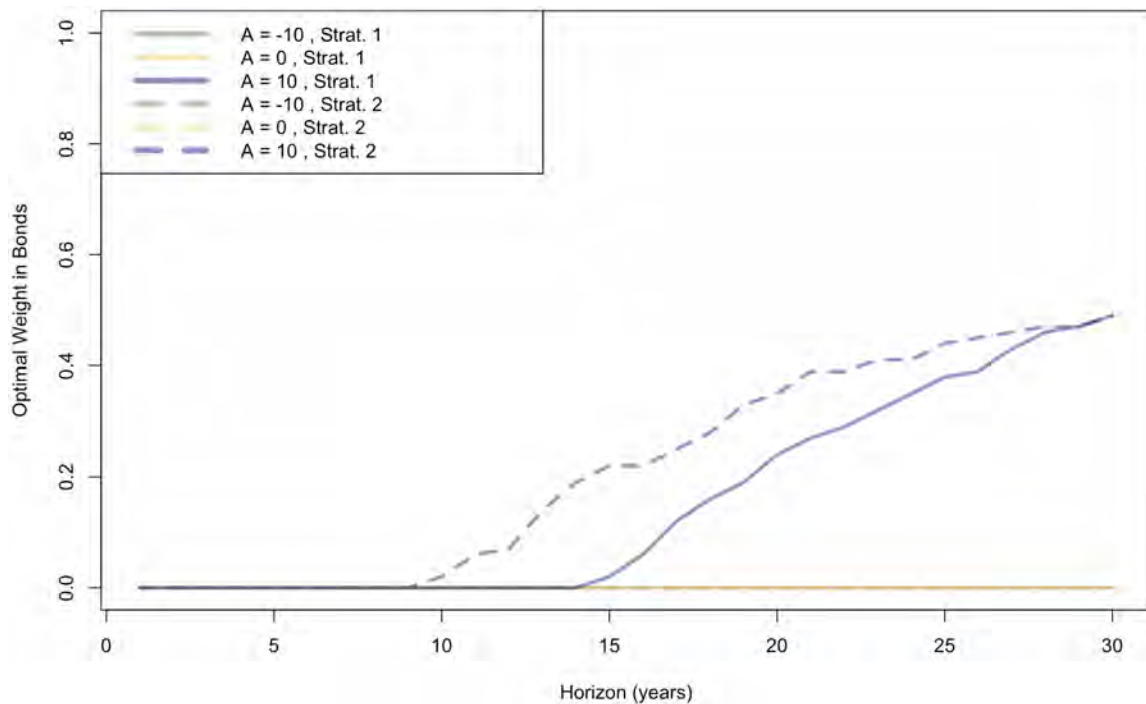

FIGURE 6.2: Optimal weight in bonds (according to utility) against the
investment horizon for both strategies.
Strategy 1: One H-year bond. Strategy 2: Multiple one-year bonds.

If the multiple one-year bond strategy is used, something similar can be observed. This is also shown in Figure 6.2. However, a risk averse investor should invest in a percentage in bonds when the investment horizon is more than nine years (instead of more than 14 years compared to the other strategy).

7.   Discussion

The goal of this research is to find the optimal long-term and short-term investment strategy for investing in bonds and stocks. This was done by creating an economic scenario generator application using R shiny. This economic scenario generator simulates stocks and interest rates in an interactive way. For the simulation of stocks, the Heston Stochastic Volatility model was used. For the simulation of interest rates, the two-Additive-Factor Gaussian model was used. Using these models, the optimal weights in bonds and stocks for different investment horizons could be found. The strategy for stocks was buy and hold. For bonds, two strategies were considered. The first was to buy a bond with a maturity equal to the investment horizon. The second strategy was to buy multiple one-year bonds until the end of the investment horizon. Two performance measures were used to investigate the optimal strategy: The Sharpe ratio and the utility function. Three different levels of risk aversion were analyzed when using the utility function.

7.1   Evaluation of the Sharpe Ratio

Figure 6.1 shows the optimal weight in bonds against the investment horizon, using the Sharpe ratio. As mentioned in chapter 6, when the horizon is short, the investor should mostly invest in bonds. As the horizon increases the investor should invest more in stocks. This phenomenon can be explained. In general, stocks have a higher return than bonds. However, stocks also have a higher risk. So when the investment horizon is short, there is not enough time to recover from potential losses (due to this higher risk) when investing in stocks. But when the horizon increases, there is time to recover from losses. Hence, stocks will then outperform bonds. This increase of investing in stocks as the horizon increases is in line with other studies (Dierkes, Erner, and Zeisberger, 2010; Veld-Merkoulova, 2011).

A difference between the two bond strategies can be observed when the investment horizon is greater than 6 years. When using strategy 1 (one H-year bond), the percentage in bonds drops to zero. When strategy 2 (multiple one-year bonds) is used, the investor should invest around 40% in bonds if the horizon is more than 6 years. This difference in strategies is due to the fact that the rates are very low (even negative) today. So if someone would buy a bond today with a maturity of H (strategy 1), he or she would not get a high return from this bond. According to the simulation of interest rates, there is a high chance for rates to go up. So if the investor would *rebalance* the bonds every year (strategy 2), the return of the bond would be higher since the rates go up.

7.2   Evaluation of the Utility Function

In Figure 6.2 the optimal weight in bonds against the investment horizon, using the utility function, can be observed. It shows that when $A = -10$ or $A = 0$, an investor should only invest in stocks regardless of the bond strategy. Using the formula for the utility function (equation 4.8), this can easily be explained. When $A = 0$, it follows that $U = \mathbb{E}[r_p]$. Since stocks have a higher return than bonds, one should only invest in stocks. When $A = -10$, the investor is a *risk seeker*. As stocks have a higher risk than bonds, the investor should only invest in stocks.
If $A = 10$ and the horizon is short, the standard deviation (σ) is low. Hence, $U = \mathbb{E}[r_p] +$ small term. So U is higher when investing stocks because of their higher return. When the horizon increases, the uncertainty increases. This means that the standard deviation increases. The right term of the equation ($\frac{\sigma_p^2 A}{2}$) will increase which will lead to a lower U. To prevent this term from increasing more, the investor should invest more in bonds since they are less risky than stocks. This will yield in a lower σ_p and thus a higher utility.

A difference between bond strategies can only be observed when $A = 10$. Using the second strategy yields in a higher percentage in bonds. This difference is because of the same reason discussed in section 7.1.

7.3   Limitations & Further Research

A limitation of this research is the fact that only **nominal** interest rates were simulated. This is the rate without taking inflation into account. The real interest rate is the nominal rate minus the inflation. A suggestion for further research would be to incorporate inflation since this would give more realistic result for the economic scenario generator. Another limitation is that the parameters were not calibrated using the most recent data. Furthermore, for the simulation of interest rates, the $\mathbb{Q}$-measure parameters were used for a $\mathbb{P}$-measure simulation. So a suggestion for further research would be to calibrate the parameters of both models under $\mathbb{P}$ using the most recent data.

A different approach for a future study could be to use different horizons for the investment strategies. For example, Figure 6.2 showed that a higher percentage of the portfolio should be invested in bonds as the horizon increased. To see if this relationship holds for a greater horizon, investigating optimal strategies for a horizon greater than 30 years could also therefore be a good suggestion for further research. Another suggestion would be to study asset allocation using a more complex portfolio (i.e. more asset than bonds or stocks).

8. Conclusion

This study has shown that the optimal short-term investment strategy, using the Sharpe ratio as performance measure, is to invest in bonds. As the horizon increases, more should be invested in stocks depending on bond strategy. Using the utility function as performance measure, the optimal short-term and long-term strategy is to invest in stocks if the investor is a risk seeker or risk neutral. When the investor is risk averse, the optimal short-term strategy is to invest in stocks and as the horizon increases, more should be invested in bonds depending on bond strategy. The study shows that if the multiple one-year bond strategy is used, a higher percentage should be invested in bonds compared to the one H-year bond strategy.

A. Instruction Manual for the Investment Tool

A large part of this research consists of making an investment tool using R Shiny. This section provides the instruction manual for this application.

A.1   Introduction to R Shiny

R Shiny is a package in R. With this package, interactive applications can be made. These applications can be published to a web page which can also be customized through *HTML* or *JavaScript*. So R Shiny allows interaction between (big) data and the web by building interactive web applications from R. The advantage of using Shiny is that the strong statistical functionalities of R can be used in combination with JavaScript without writing code in JavaScript.

Concerning this paper, this application allows users to visualize the simulations and portfolio results. Users can interact with the application by changing the model parameters and specifying investment decisions. The application consists of two main parts: **Model parameters** and **results**.

A.2   Model Parameters

When the application is started, the user will see the following screen (Figure A.1).
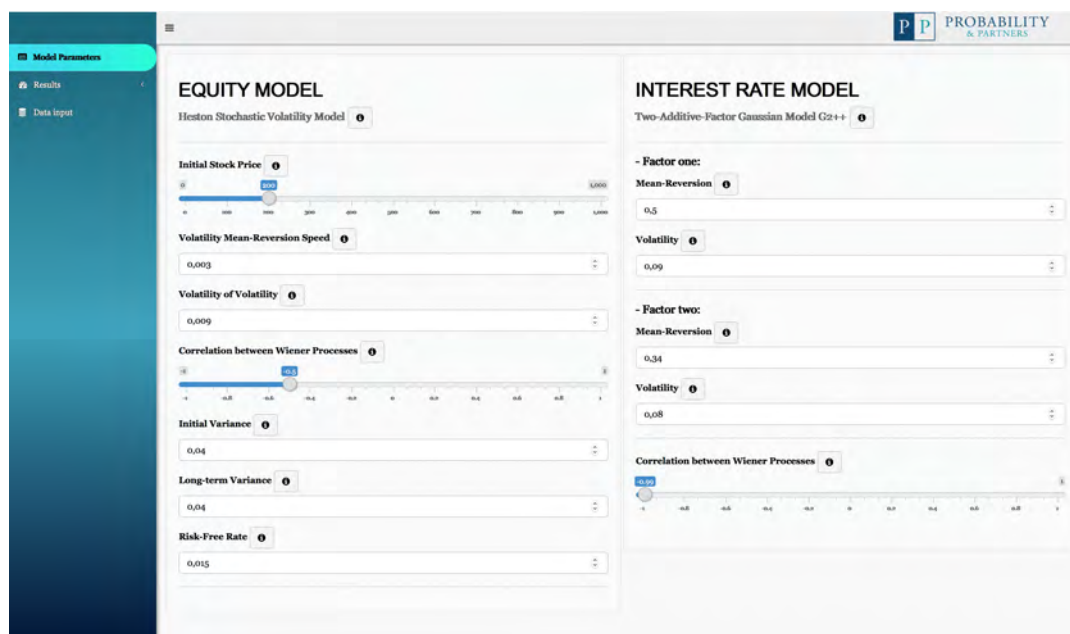

FIGURE A.1: Model Parameters tab of the application.

On the left side of the application there are multiple tabs where users can choose from. When the tool is started, the *Model Parameters* tab is automatically opened. This is the part where users can adjust the model parameters. They can adjust the parameters of the **stocks** model and the **interest rate** model.

A.2.1 Stocks

As mentioned in Chapter 2, the Heston Stochastic Volatility Model is used for modeling Stocks. This application allows the user to change the initial stock price and the parameters of the Heston model which are mentioned in Section 2.3. At each parameter there is an 'information' button (Figure A.2) which contains information about the model parameter.

FIGURE A.2: Information button.

A.2.2 Interest Rate

As discussed in Chapter 3, the Two-Additive-Factor Gaussian Model G2++ is used for modeling interest rate. Similar as for stocks, the application allows the user the adjust the parameters of the used model with corresponding information buttons (Figure A.2).

A.3 Results

If the user clicks on the *Results* tab on the left side of the application, the user has two options to click on: **Simulation** and **investment**.

A.3.1 Simulation Results

On the left side of the *Simulation Results* page, the user can adjust the simulation parameters (Figure A.3): The sampling frequency, the number of simulations, the investment horizon and the maturity.
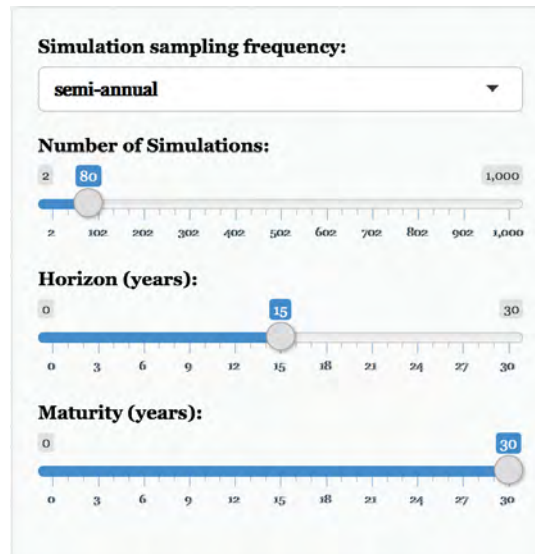
FIGURE A.3: Simulation parameters.

On the right side of the page, the user can see the simulation results resulting from the selected model and simulation parameters. From top the bottom, the user sees the following simulation plots over time:

- Stock price

- Log returns

- Volatility

- Short rate

- Yield Curve

An example of the first four plots are shown in Figure A.4. The short rate plot is an interactive plot where the user can click on a point in time (bottom figure). This results in the corresponding yield curve at that time with the selected maturity. An example is shown in Figure
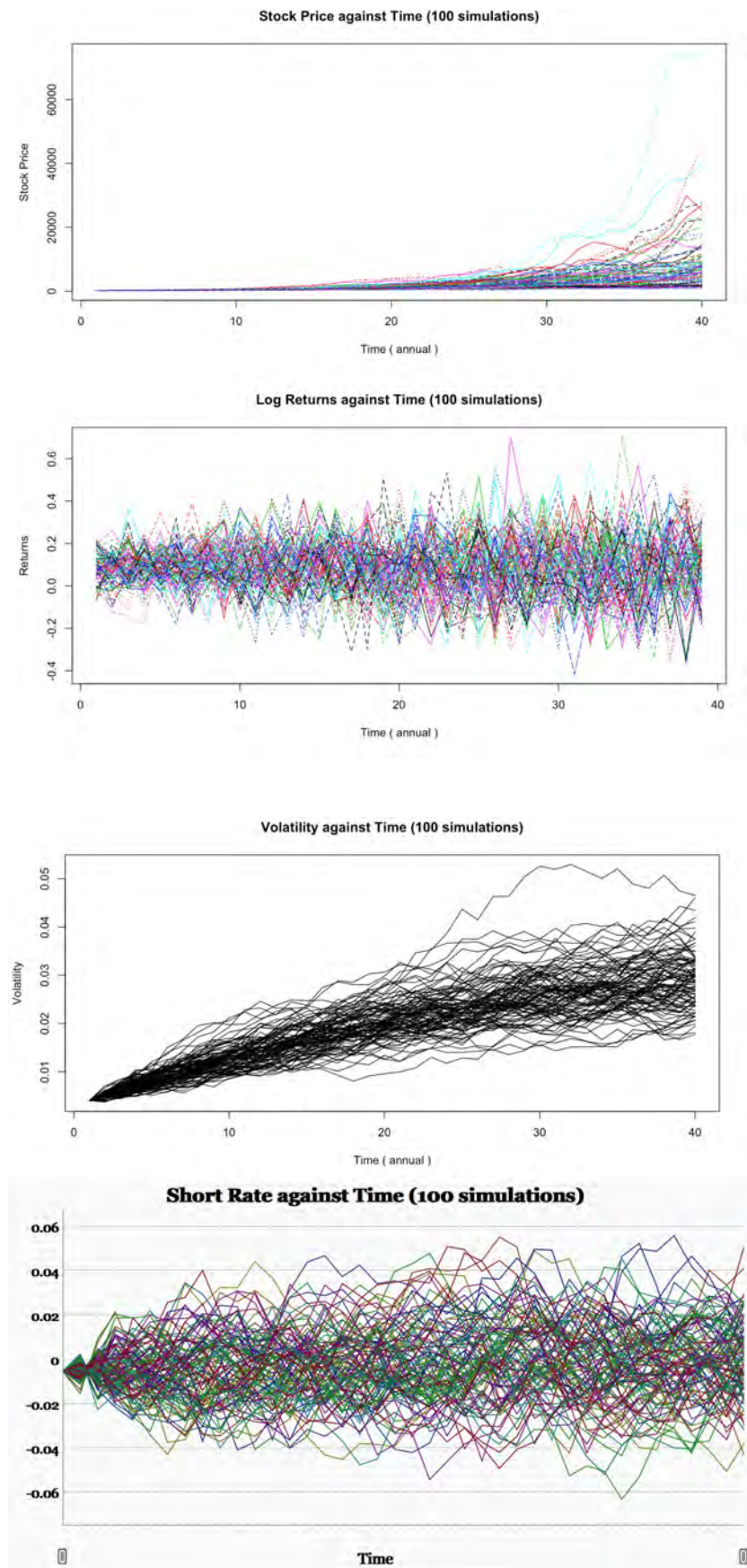
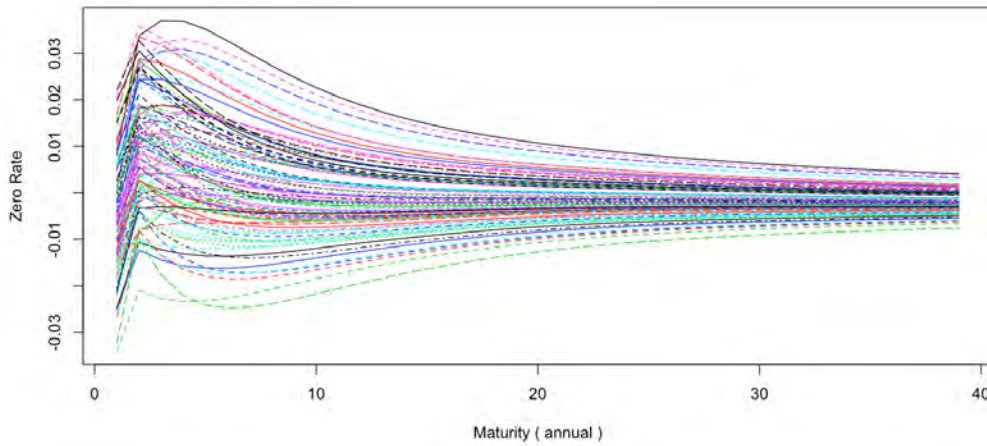FIGURE A.4: Example of simulation plots.

FIGURE A.5: Example of interactive simulated yield curves.

A.3.2 Investment Results

This page is divided into four parts. **Variables**, **Plots**, **Statistics** & **Investment table**.

Variables

In this section, one can change the investment variables:

- Weight in bonds (= 1- weight in stocks)

- Maturity of the bond(s)

- Horizon

- Risk aversion (between -10 and 10)

Plots

In this section four plots against the weight in bonds of the selected strategy is shown. The top left plot shows the expect return, the top right plot the standard deviation, the bottom left shows the Sharpe ratio and the bottom right shows the utility. All plots contain a red vertical line which represents the weight in bonds selected at the *variables* section. An example of these plots are shown in A.6
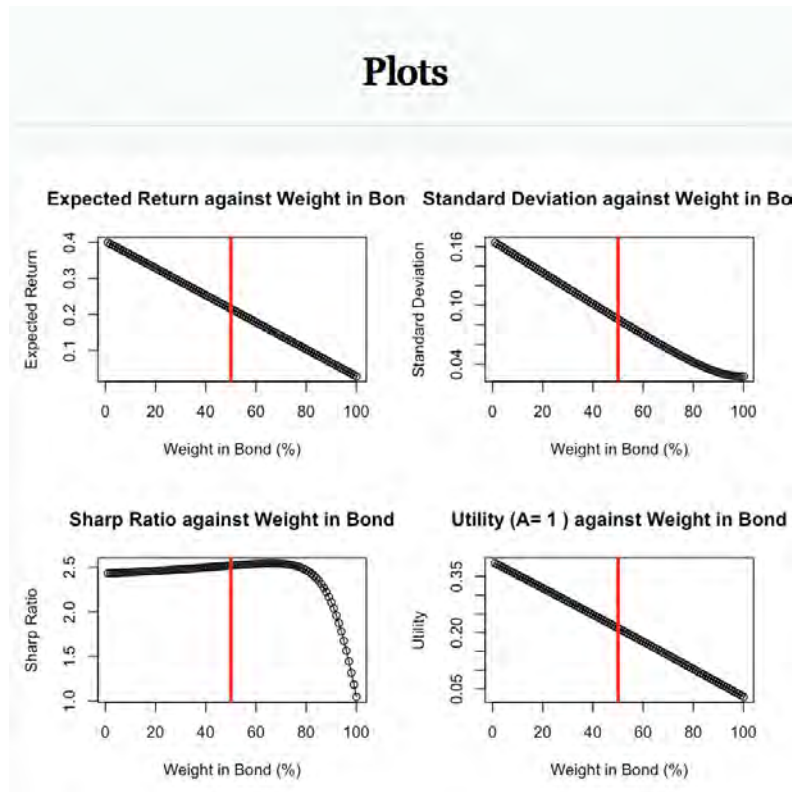
FIGURE A.6: Example of the four plots shown for a certain strategy. The red line represents the selected weight in bonds.

Statistics

This section shows some statistics of the selected strategy. From top to bottom it shows:

- Expected return

- Minimum return

- Maximum return

- Standard deviation of the return

- Sharpe ratio

- Amount of simulations

- Sampling frequency

- Utility

- Optimal weight in bonds using utility

- Optimal weight in bonds using the Sharpe Ratio

Below these statistics, the amount of decimals can be adjusted.

Investment Table

The statistics can be added to a table on the top of the screen using the **Add Strategy to Table** button on the right side of the page. In this table all the statistics of the added strategy are shown. Each row contains a strategy. A strategy can be deleted from the table by clicking the *delete* button on the most right column of the table. Below this button, there is also a *distribution* button which will show the distribution of the returns of the strategy. Two or more distributions of the strategies of the table can be compared by selecting the strategies and clicking the **Compare selected strategies** button on the right side of the page. Below this button can also click the **Download the data** button which will download the investment table to *.csv* format.

B. Derivations

B.1 The exact solution of the SDE for an Ornstein-Uhlenbeck process

We start with the SDE:

$$dx(t) = -ax(t)dt + \sigma dW_x(t)$$

We want to seperate all the $x(t)$-terms and bring them to the left side.

$$dx(t) + ax(t)dt = \sigma dW_x(t)$$

Using the method of *Integrating Factors* with the factor e^{at}, we get

$$e^{at}(dx(t) + ax(t)dt) = e^{at}\sigma dW_x(t)$$

$$e^{at}dx(t) + ae^{at}x(t)dt = e^{at}\sigma dW_x(t)$$

One can observe that the left side of the equation can be written as:

$$dXY + XdY = d(XY),$$

$$Y = e^{at}, \qquad X = x(t), \qquad dY = ae^{at}, \qquad dX = dx(t).$$

In other words, the left side of the equation can be written as the backwards of the chain-rule into the form $d(XY)$ as

$$d(e^{at}x(t)) = e^{at}\sigma dW(t)$$

This way, there is only one $x(t)$ term left and the right side of the equation does not depend on $x(t)$ anymore. The next step is to write the stochastic differential equation back to its integral form.

$$\int_t^T d(e^{as}x(s)) = \int_t^T \sigma e^{as}dW(s)$$

Then we can simplify the equation by solving the integral on the left side of the equation.

$$e^{aT}x(T) - e^{at}x(t) = \int_t^T \sigma e^{as}dW(s)$$

$$e^{aT}x(T) = e^{at}x(t) + \int_t^T \sigma e^{as}dW(s)$$

$$x(T) = e^{-a(T-t)}x(t) + e^{-aT}\int_t^T \sigma e^{as}dW(s).$$

The σ-term can be taken out of the integral, since it is a constant. Furthermore, e^{-aT} can be put inside of the integral for ease of notation.

$$x(T) = e^{-a(T-t)}x(t) + \sigma \int_t^T e^{-a(T-s)}dW(s)$$

The last step is to solve the *Itô integral* of the form $\int_a^b g(s)dW(s)$ by writing the stochastic integral in terms of its first and second moment. In other words, the *Itô integral* will be written in terms of a standard Gaussian variable, $Z_x \sim N(0,1)$ that can be used to simulate the SDE for factor x.

First let us look into the first moment of the stochastic differential equation (SDE) for factor x, where we use the fact that expected value of a stochastic integral (Wiener process), $\mathbb{E}[\int_t^T g(s)dW(s)]$, is zero.

$$\mathbb{E}\left[x(T)\right] = \mathbb{E}\left[e^{-a(T-t)}x(t)\right] + \mathbb{E}\left[\sigma \int_t^T e^{-a(T-s)}dW(s)\right]$$

$$\mathbb{E}\left[x(T)\right] = e^{-a(T-t)}x(t) + 0$$

So effectively, the expected value of factor x is equal to the drift term, $e^{-a(T-t)}x(t)$. For the second moment of factor x, one can leave out the drift term, since this is a deterministic term and therefore has a variance equal to zero. The variance of the stochastic part can be decomposed into two terms:

$$\mathbb{V}ar[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\mathbb{V}ar\left[\sigma \int_t^T e^{-a(T-s)}dW(s)\right] = \mathbb{E}\left[\left(\sigma \int_t^T e^{-a(T-s)}dW(s)\right)^2\right] - \left(\mathbb{E}\left[\sigma \int_t^T e^{-a(T-s)}dW(s)\right]\right)^2$$

It has already been shown that $\mathbb{E}\left[\sigma \int_t^T e^{-a(T-s)}dW(s)\right] = 0$, which leaves us with

$$\mathbb{V}ar\left[\sigma \int_t^T e^{-a(T-s)}dW(s)\right] = \mathbb{E}\left[\left(\sigma \int_t^T e^{-a(T-s)}dW(s)\right)^2\right].$$

This term can be solved with the *Îto Isometry* rule:

$$\mathbb{E}\left[\left(\int_t^T g(s)dW(s)\right)^2\right] = \mathbb{E}\left[\int_t^T g(s)^2 ds\right].$$

The variance of factor x then boils down to

$$\mathbb{V}ar\left[\sigma^2\int_t^T e^{-a(T-s)}dW(s)\right] = \mathbb{E}\left[\sigma^2\int_t^T \left(e^{-a(T-s)}\right)^2 ds\right],$$

$$= \mathbb{E}\left[\sigma^2\int_t^T e^{-2a(T-s)}ds\right],$$

$$= \mathbb{E}\left[\sigma^2\left[\frac{e^{-2a(T-s)}}{2a}\right]_t^T\right],$$

$$= \mathbb{E}\left[\sigma^2\left(\frac{e^{-2a(T-T)}}{2a}\right) - \sigma^2\left(\frac{e^{-2a(T-t)}}{2a}\right)\right],$$

$$= \mathbb{E}\left[\sigma^2\left(\frac{1}{2a}\right) - \sigma^2\left(\frac{e^{-2a(T-t)}}{2a}\right)\right],$$

$$= \mathbb{E}\left[\sigma^2\left(\frac{1-e^{-2a(T-t)}}{2a}\right)\right],$$

$$\mathbb{V}ar\left[\sigma^2\int_t^T e^{-a(T-s)}dW(s)\right] = \sigma^2\left(\frac{1-e^{-2a(T-t)}}{2a}\right).$$

By writing factor x in terms of the expectation of x and variance of x, the final equation for factor x can be written as:

$$x(T) = e^{-a(T-t)}x(t) + \sigma\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}Z_x(t),$$

where $Z_x(t) \sim N(0,1)$

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