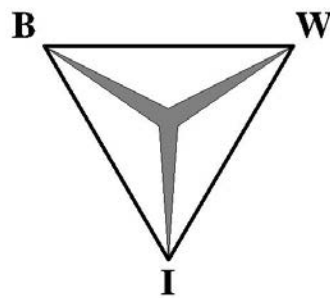*vrije* Universiteit *amsterdam*

Business Mathematics and Informatics

Numerical solution of modified Black-Scholes equation pricing stock options with discrete dividend

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BMI Thesis

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Preface

BMI Thesis is the one component of acquiring the Master's degree in Business Mathematics and Informatics. Business Mathematics and Informatics (BMI) is a multi-disciplinary study that is focussed on the integration of Business Economics, Mathematical and Informatics principles with the aim of solving management and operational problems in the industry with a quantitative thrust.

This BMI thesis starts with the brief introduction of option: the most essential derivatives in financial markets. Then Black Scholes model is described to price the option. Based on Black Scholes model, we are more interested in practical problems: modified Black Scholes model with discrete dividend payments. Theoretical solutions and numerical experiments are given in the end.

After reading this thesis, one is able to know some details on option pricing, Black Scholes model and how to solve the Black Scholes equation.

I would like to thank my supervisor Professor Dr. André Ran for his help, support and his always enthusiastic feedback!

Ermo Shen
Amstelveen, 2008

Summary

This paper focuses on the numerical solution of the modified Black-Scholes equation with discrete dividend. We use the Dirac delta function to model the valuation of stock options with discrete dividend payments. Explicit solution is obtained by applying the Mellin transform to the modified Black-Schole equation. Numerical quadrature approximations and illustrative examples are given in the end.

Contents

Chapter 1

Introduction

In option pricing theory, the Black-Scholes equation is one of the most effective models. For European options, the Black-Scholes equation can be solved in terms of a diffusion equation boundary value problem [2], or directly using the Mellin transform [3, 4]. There are two ways to solve the option pricing problem: analytical approaches and numerical approaches. For European options with continuous payoff functions, the analytical solution is relatively easy to obtain [2, 3]. However, finding an expression for the solution of the Black-Scholes equation when coefficients are discontinuous ordinary functions or generalized functions is not an easy matter.

The Black-Scholes model for pricing stock options when there are dividend payments $D(S, t)$ is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D(S,t))\frac{\partial V}{\partial S} - rV = 0, \qquad 0 < S < \infty,\ 0 < t < T.$$

If a discrete dividend yield, independent of S with dividend date t_d, is considered, then $D(S, t)$ takes the form

$$D(S,t) = A\delta(t - t_d)S, \qquad 0 < t_d < T \qquad (1.1)$$

where A is a constant and $\delta(t - t_d)$ is the shifted Dirac delta function (see [2]).It is well known that $\delta(x)$ is not an ordinary function, but this generalized function can be obtained as the limit of special sequences of ordinary functions. Such a discrete dividend payment inevitably results in a jump in the value of the underlying asset across the dividend date. Using financial arguments, it can be

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shown that the effect of this discontinuous change in the value of the asset of a contingent option on the asset is the jump condition,

$$V(S, t_d^-) = V(Se^{-A}, t_d^+), \tag{1.2}$$

where t_d^- and t_d^+ denote just before and just after the dividend payment, respectively (see[2]).

This paper deals with the construction of numerical solutions of modified Black-Scholes equations of the type

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - A\delta(t - t_d))S\frac{\partial V}{\partial S} - rV = 0, \tag{1.3}$$

$$V(S, T) = f(S), \qquad 0 < S < \infty, \ 0 < t_d < T, \ 0 < t < T. \tag{1.4}$$

An ordinary function $V(S, t)$ is said to be a financially admissible solution of problem (3) and (4) if, for $t \neq t_d$, $V(S, t)$ satisfies the Black-Scholes equation without dividend payment,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.5}$$

as well as the final condition

$$\lim_{t \to T^-} V(S, t) = f(S), \tag{1.6}$$

almost everywhere for S and for every S at which f is continuous, and the jump condition (1.2) is

$$\lim_{t \to t_d^-} V(S, t) = \lim_{t \to t_d^+} V(Se^{-A}, t). \tag{1.7}$$

This paper is organized as follows. Chapter 2 gives a brief description of option and its pricing model: the Black-Scholes model with several basic assumptions.

Chapter 3 requires some more advanced mathematics. In this chapter, we will introduce Fourier and Mellin transform as well as the approximation of the generalized function $\delta(t - t_d)$ by means of an ordinary function sequence $f_n(t)$. We also provide the solution of the approximation problem

$$\left. \begin{aligned} &\frac{\partial V_n}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_n}{\partial S^2} + (r - Af_n(t))S\frac{\partial V_n}{\partial S} - rV_n = 0, \\ &0 < S < \infty, \ 0 < t < T \\ &V_n(S, T) = f(S), \qquad 0 < S < \infty \end{aligned} \right\} \tag{1.8}$$

using the Mellin transform technique developed in [3]. It is proved that $\{V_n(S,t)\}$ is pointwise convergent to a financially admissible solution $V(S,t)$ of problem (3) and (4) that is explicitly expressed in terms of the payoff function, the dividend yield A, the volatility σ, and the interest rate r.

Chapter 4 is concerned with the numerical approximation of $V(S,t)$. Two different numerical quadrature schemes will be applied to the approximation of the integral solution: Simpson and Guass-Hermite schemes.

Chapter 2

Option and its price

One of the most significant developments in financial markets in recent years has been the growth of futures, options, and related derivatives markets. These instruments provide payoffs that depend on the values of other assets such as commodity prices, bond and stock prices, or market index values. For this reason these instruments sometimes are called derivative assets, or contingent claims. Their values derive from or are contingent on the values of other assets.

2.1 What is an option?

The simplest financial option, a **European call option**, is a contract with the following conditions:

- At a prescribed time in the future, known as the **expiry date** or **expiration date**, the holder of the option *may*

- purchase a prescribed asset, known as the **underlying asset** or, briefly, the **underlying**, for a

- prescribed amount, known as the **exercise price** or **strike price**

The word 'may' in this description implies that for the holder of the option, this contract is a '*right*' and not an '*obligation*'. The other party to the contract, who is known as the **writer**, does have a potential obligation: he *must* sell the asset if the holder chooses to buy it. Since the option confers on its holder a right with no obligation it has some value. Moreover, it must be paid at the time of the opening

of the contract. Conversely, the writer of the option must be compensated for the obligation he has assumed.

European Put Options The option to buy an asset discussed above is known as a **call** option. The right to sell an asset is known as a **put option** and has payoff properties which are opposite to those of a call. A put option allows its holder to sell the asset on a certain date for a prescribed amount. The writer is then obliged to buy the asset. Whereas the holder of a call option wants the asset price to rise – the higher the asset price at expiry the greater the profit – the holder of a put option wants the asset price to fall as low as possible. The value of a put option also increases with the exercise price, since with a higher exercise price more is received for the asset expiry.

American Options Above we described the European call/put options, but nowadays most options are what is called American. The European/American classification has nothing to do with the continent of origin but refers to a technicality in the option contract. An **American option** is one that may be exercised at *any time* prior to expiry, whereas the European options may only be exercised at expiry.

2.2   Interest rate and present value

In this paper, we assume that the interest rate is a known function of time, not necessarily constant. This is not an unreasonable assumption when valuing options, since a typical equity option has a total lifespan of about nine months.

For valuing options the most important concept concerning interest rate is that of **present value** or **discounting**, i.e., how much would I pay now to receive a guaranteed amount E at the future time T?

If we assume the interest rates are constant, the answer to this question is found by discounting the future value, E, using continuously compounded interest. With a constant interest rate r, money in the bank $M(t)$ grows exponentially according to

$$\frac{dM}{M} = r\,dt. \tag{2.1}$$

The solution of this is simply

$$M = ce^{rt},$$

where c is the constant of integration. Since $M = E$ at $t = T$, the value at time t of the certain payoff is

$$M = Ee^{-(T-t)}.$$

If interest rates are a known function of time $r(t)$, then (2.1) can be modified trivially and results in

$$M = Ee^{-\int_t^T r(s)\,ds}.$$

2.3 Black-Scholes model

2.3.1 Asset price random walks

It is often stated that asset prices must move randomly because of the efficient market hypothesis. There are several different forms of this hypothesis with different assumptions, but they all basically say two things:

- The past history is fully reflected in the present price, which does not hold any further information.

- Markets respond immediately to any new information about an asset.

Thus the modelling of an asset is really about modelling the arrival of new information which affects the price. With the two assumptions above, unanticipated changes in the asset price are a Markov process.

Now suppose that at time t the asset price is S. Let us consider a small subsequent time interval dt, during which S changes to $S + dS$. The commonest model decomposes this return into two parts. One is a predictable, deterministic and anticipated return akin to the return on money invested in a risk-free bank. It gives a contribution

$$\mu\,dt$$

to the return dS/S, where μ is a measure of the average rate of growth of the asset price, also known as the drift. In simple models μ is taken to be a constant. In more complicated models, for exchange rates, for example, μ can be a function of S and t.

The second contribution to dS/S models the random change in the asset price in response to external effects, such as unexpected news. It is represented by a

random sample drawn from a normal distribution with mean zero and adds a term

$$\sigma \, dX$$

to dS/S. Here σ is a number called the **volatility**, which measures the standard deviation of the returns. The quantity dX is the sample from a normal distribution, which is discussed further below.

Putting these contributions together, we obtain the stochastic differential equation

$$\frac{dS}{S} = \sigma \, dX + \mu \, dt, \tag{2.2}$$

which is the mathematical representation of our simple recipe for generating asset prices.

The term dX, which contains the randomness that is certainly a feature of asset prices, is known as a **Wiener process**. It has the following properties:

- dX is a random variable, drawn from a normal distribution, independent of the value of X on the history of X;

- the mean of dX is zero;

- the variance of dX is dt.

2.3.2   Arbitrage

One of the fundamental concepts underlying the theory of financial derivative pricing and hedging is that of **arbitrage**. This can be loosely stated as 'there's no such thing as a free lunch.' More formally, in financial terms, there are never any opportunities to make an instantaneous risk-free profit.

2.3.3   Option values

There are some simple notation which we will use:

- We denote by V the value of an option; when the distinction is important we use $C(S,t)$ and $P(S,t)$ to denote a call and a put respectively. This value is a function of the current value of the underlying asset, S and time, t: $V = V(S,t)$. The value of the option also depends on the following parameters:

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- σ, the volatility of the underlying asset;

- E, the exercise price;

- T, the expiry date; r, the interest rate.

First, consider what happens just at the moment of expiry of a call option, that is, at time $t = T$. A simple arbitrage argument tells us its value at this special time.

If $S > E$ at expiry, it makes financial sense to exercise the call option, handing over an amount E, to obtain an asset worth S. The profit from such a transaction is then $S - E$. On the other hand, if $S < E$ at expiry, we should not exercise the option because we would make a loss of $E - S$. In this case, the option expires worthless. Thus, the value of the call option at expiry can be written as

$$C(S, T) = \max(S - E, 0). \tag{2.3}$$

As we get nearer to the expiry date we can expect the value of our call option to approach (2.3).

2.3.4 Black-Scholes analysis

Before describing the Black-Scholes analysis which leads to the value of an option we list the assumptions that we make:

- The asset price follows the lognormal random walk (2.2).

- The risk-free interest rate r and the asset volatility σ are known functions of time over the life of the option.

- There are no transaction costs associated with hedging a portfolio.

- The underlying asset pays no dividends during the life of the option. We will drop this assumption in further analysis.

- There are no arbitrage possibilities.

- Trading of the underlying asset can take place continuously.

- Short selling is permitted and the asset are divisible.

Suppose that we have an option whose value $V(S, t)$ depends only on S and t. It is not necessary at this stage to specify whether V is a call or a put; indeed, V can be the value of a whole portfolio of different options although for simplicity we can think of a simple call or put. Using Itô's lemma, we can write

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \qquad (2.4)$$

This give the random walk followed by V. Note that we require V to have at least one t derivative and two S derivatives.

Now construct a portfolio consisting of one option and a number $-\Delta$ of the underlying asset. This number is as yet unspecified. The value of this portfolio is

$$\Pi = V - \Delta S. \qquad (2.5)$$

The jump in the value of this portfolio in one time-step is

$$d\Pi = dV - \Delta \, dS.$$

Here Δ is held fixed during the time-step; if it were not then $d\Pi$ would contain terms in $d\Delta$. Putting (2.2), (2.4), (2.5) together, we find that Π follows the random walk

$$d\Pi = \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt. \qquad (2.6)$$

We can eliminate the random component in this random walk by choosing

$$\Delta = \frac{\partial V}{\partial S}. \qquad (2.7)$$

Note that Δ is the value of $\partial V / \partial S$ at the start of the time-step dt.

This results in a portfolio whose increment is wholly deterministic:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \qquad (2.8)$$

We now appeal to the concepts of arbitrage and supply and demand, with the assumption of no transaction costs. The return on an amount Π invested in

riskless assets would see a growth of $r\Pi\,dt$ in a time dt. If the right hand side of (2.8) were greater than this amount, an arbitrager could make a guaranteed riskless portfolio. The return for this risk-free strategy would be greater than the cost of borrowing. Conversely, if the right-hand side of (2.8) were less than $r\Pi\,dt$ then the arbitrager would short the portfolio and invest Π in the bank. Either way the arbitrager would make a riskless, no cost, instantaneous profit. The existence of such arbitragers with the ability to trade at low cost ensures that the return on the portfolio and on the riskless asset are equal. Thus, we have

$$r\Pi\,dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt. \tag{2.9}$$

Substituting (2.5) and (2.7) into (2.9) and dividing throughout by dt we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.10}$$

This is the **Black-Scholes partial differential equation**.

2.4 Black-Scholes model with discrete dividend

2.4.1 Dividend structures

Many assets, such as equities, pay out **dividends**. These are payments to shareholders out of the profits made by the company concerned, and the likely future dividend stream of a company is reflected in today's share price. The price of an option on an underlying asset that pays dividends is affected by the payments, so we must modify the Black-Scholes analysis.

There are several possible different structures for dividend payments. Individual companies usually make two or four payments per year, which may need to be treated discretely, but the large number of dividend payments on an index such as S&P 500 are so frequent that it may be best to regard them as a continuous payment rather than as a succession of discrete payments.

2.4.2 A constant dividend yield

Suppose that in a time dt the underlying asset pays out a dividend $D_0 S dt$ where D_0 is a constant. This payment is independent of time except through the dependence on S. The **dividend yield** is defined as the proportion of the asset

price paid out per unit time in this way. Thus the dividend $D_0 S dt$ represents a constant and continuous dividend yield D_0.

First, we consider the effect of the dividend payments on the asset price. Arbitrage considerations show that in each time step dt, the asset price must fall by the amount of the dividend payment, $D_0 dt$, in addition to the usual fluctuation. It follows that the random walk for the asset price (2.2) is modified to

$$dS = \sigma S \, dX + (\mu - D_0) S \, dt \qquad (2.11)$$

Since we receive $D_0 S dt$ for every asset held and since we hold $-\Delta$ of the underlying, our portfolio changes by an amount

$$-D_0 S \Delta dt, \qquad (2.12)$$

i.e., by the dividend our asset pays. Thus, we must add (2.12) to our earlier $d\Pi$ to arrive at

$$d\Pi = dV - \Delta dS - D_0 S \Delta dt.$$

The analysis proceeds exactly as before but with the addition of this new term. We find that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0. \qquad (2.13)$$

For a call option the final condition is still $C(S,T) = \max(S - E, 0)$, and the boundary condition at $S = 0$ remains as $C(0,t) = 0$. The only change to the boundary conditions when we use the modified Black-Scholes equation (2.13) is that

$$C(S,t) \sim S e^{-D_0(T-t)} \quad \text{as} \quad S \to \infty. \qquad (2.14)$$

2.4.3   Discrete dividend payments

Suppose that our asset pays just one dividend during the lifetime of the option, at time $t = t_d$. As above, we shall consider only the case in which the dividend yield is a known constant d_y. Thus, at time t_d, the holder of the asset receives a payment $d_y S$, where S is the asset price just before the dividend is paid.

Consider the effect of the dividend payment on the asset price. Its value just before the dividend time t_d^-, cannot equal is value just after, at time t_d^+. If it did, the strategy of buying the asset immediately before t_d, collecting the dividend, and selling straight away, would yield a risk-free profit. It is clear that, in the

absence of other factors such as taxes, the asset price must fall by exactly the amount of the dividend payment. Thus

$$S(t_d^+) = S(t_d^-) - d_y S(t_d^-) = S(t_d^-)(1 - d_y). \tag{2.15}$$

The discrete dividend payment results in a jump in the value of the underlying asset across the dividend date. To model the jump condition, we have to introduce the Dirac delta function:

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \tag{2.16}$$

where

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$

The constant yield has $D(S,t) = D_0 S$ while the discrete case, $D(S,t) = D_\delta S \delta(t - t_d)$ for some constant D_δ. So the Black-Scholes model of discrete dividend payments now becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D(S,t))\frac{\partial V}{\partial S} - rV = 0, \quad 0 < S < \infty, \ 0 < t < T. \tag{2.17}$$

where

$$D(S,t) = A\delta(t - t_d)S, \quad 0 < t_d < T.$$

Here A is a constant and $\delta(t - t_d)$ is the shifted Dirac delta function.

Chapter 3

Integral solution

3.1 Mellin transform

For the sake of clarity in the presentation, we recall some notation and results about integral transforms and generalized functions. We denote by L^1 the set of all Lebesgue integrable functions in $\mathbb{R}$. If f and g belong to L^1, then the convolution of f and g given by the function

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy, \tag{3.1}$$

exists for almost every x and $f * g \in L^1$ (see [6], p,232). If $f \in L^1, g \in C^k$ and $D^\alpha g$ is bounded, then, by ([6],p.233), $f * g \in C^k$ and

$$D^\alpha(f * g) = f * (D^\alpha g), \qquad \alpha \leqslant k. \tag{3.2}$$

The Fourier transform of $f \in L^1$ is defined by

$$\mathcal{F}[f(x)] = F(\omega) = \int_{\infty}^{\infty} f(x)e^{-i\omega x}\, dx. \tag{3.3}$$

Let $G(\omega)$ be the Fourier transform of $g(x)$. If $G \in L^1$, it's inverse Fourier transform is defined by

$$\mathcal{F}^{-1}[G(\omega)](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)e^{i\omega x}d\omega, \tag{3.4}$$

and from ([6],p.243) it holds that $\mathcal{F}^{-1}[G(\omega)](x) = g(x)$ almost everywhere. Let $f, g \in L^1$ with Fourier transform F and G respectively, then from ([6],p.241),

$$\mathcal{F}[f * g] = F(\omega)G(\omega). \tag{3.5}$$

From the properties of the Fourier transform, it is well known that (see[7])

$$\mathcal{F}\left[e^{-a^2 x^2}\right](\omega) = \frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}}, \qquad a > 0, \tag{3.6}$$

and

$$\mathcal{F}^{-1}\left[e^{ib\omega} F(\omega)\right](x) = \mathcal{F}^{-1}[F(\omega)](x + b). \tag{3.7}$$

Let f be a real function defined on $(0, \infty)$. The **Mellin transform** of f is the complex valued function defined by

$$M[f(x)] = f^*(z) = \int_0^\infty f(x) x^{z-1} \, dx, \qquad z = \alpha + i\omega \tag{3.8}$$

assuming the integral exists. For $\eta, \nu \in \mathbb{R}$, with $\eta < \nu$, we define the set $\mathcal{M}(\eta, \nu)$ as follows:

$$\mathcal{M}(\eta, \nu) = \left\{ f : (0, \infty) \to \mathbb{R} \,\middle|\, \int_0^\infty x^{\alpha-1} |f(x)| dx < \infty, \eta < \alpha < \nu \right\}.$$

It is easy to show that if $f \in \mathcal{M}(\eta, \nu)$, then $M[f(x)](z)$ exists on the strip $\langle \eta, \nu \rangle = \{ z = \alpha + i\omega : \eta < \alpha < \nu, \omega \in \mathbb{R} \}$. Furthermore, $f \in \mathcal{M}(\eta, \nu)$ if and only if $e^{-\alpha l} f(e^{-l}) \in L^1$, for $\eta < \alpha < \nu$.

If $f \in \mathcal{M}(\eta, \nu)$ and its Mellin transform $f^*(\alpha + i\omega)$ lies in L^1 with respect to ω, for each fixed α such that $\eta < \alpha < \nu$, the **inverse Mellin transform** of f^* is defined by

$$M^{-1}[f^*](x) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} x^{-z} f^*(z) dz, \tag{3.9}$$

it holds that $M^{-1}[f^*(z)] = f(x)$ a.e.

Let f be a function belonging to $\mathcal{M}(\eta, \nu)$ with $f(x) \in C^2(\mathbb{R})$ such that, for $\eta < \alpha < \nu$,

$$\begin{aligned} \lim_{x \to 0^+} x^\alpha f(x) &= \lim_{x \to \infty} x^\alpha f(x) = 0, \\ \lim_{x \to 0^+} x^{\alpha+1} f'(x) &= \lim_{x \to \infty} x^{\alpha+1} f'(x) = 0, \end{aligned} \tag{3.10}$$

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Then, for all $z \in \langle \eta, \nu \rangle$ one gets

$$M[xf'(x)](z) = -zM[f(x)](z),$$
$$M[x^2 f''(x)](z) = (z^2 + z)M[f(x)](z). \tag{3.11}$$

For instance, if f is a locally integrable function satisfying $f(x) = O(x^{-\eta})$ when $x \to 0$ and $f(x) = O(x^{-\nu})$ when $x \to \infty$, then the integral (3.8) converges for all $z \in \langle \eta, \nu \rangle$, and defines an analytic function in this interval.

The Mellin transform and the Fourier transform and their inverses are related by the following formulas. For $f \in \mathcal{M}(\eta, \nu)$,

$$M[f(x)](z) = \mathcal{F}[e^{-\alpha x} f(e^{-x})](\omega), \qquad z = \alpha + i\omega \in \langle \eta, \nu \rangle. \tag{3.12}$$

Furthermore, if $f^*(z) \in L^1$, one gets

$$M^{-1}[f^*(z)](x) = x^{-\alpha} \mathcal{F}^{-1}[f^*(z)](-\ln x). \tag{3.13}$$

We denote by K the space of functions $\varphi : \mathbb{R} \to \mathbb{R}$ in $C^\infty(\mathbb{R})$ having a compact support. A generalized function f is defined as a continuous linear functional on K, and we denote $f(\varphi) = (f, \varphi)$ (see [5]). The space of all generalized functions on K will be called K'. We said that $f(x)$ is an ordinary function if $f \in L^1[a, b]$ for all $a < b$. With each ordinary function $f(x)$, there is associated a continuous linear functional on the space K through

$$(f, \varphi) = \int_\infty^\infty f(t)\varphi(t)\, dt.$$

The Dirac delta function is defined as the generalized function which assigns value $\varphi(0)$ to each function $\varphi(x) \in K$, i.e., $(\delta, \varphi) = \varphi(0)$. Note that she shifted Dirac delta function $\delta(t - t_d)$ acts on K in the form $(\delta(t - t_d), \varphi(t)) = \varphi(t_d)$; see ([5],pp,11-13)

A sequence of generalized functions $f_1, f_2, \ldots, f_n, \ldots$ converges in K' to the generalized function f if, for all $\varphi \in K$ (see [5],p.63),

$$(f, \varphi) = \lim_{n \to \infty} (f_n, \varphi).$$

Now we introduce a class of sequences of ordinary functions having a particular interest in practical applications.

A sequence of ordinary functions $\{f_n(t)\}$ is said to be nice shifted delta-defining if, for any interval I_0, the quantities

$$\left| \int_I f_n(t)dt \right|, \qquad I \subset I_0$$

are bounded by a constant depending on neither I nor n, and if

$$\lim_{n \to \infty} \int_I f_n(t)dt = \begin{cases} 0, & \text{when } t_d \text{ is exterior to } I, \\ 1, & \text{when } t_d \text{ is interior to } I. \end{cases} \tag{3.14}$$

3.2 Integral solution of the approximate problem

This section deals with the construction of a formal solution of the approximate problem (1.8), where $\{f_n(t)\}$ is an arbitrary nice shifted delta-defining sequence. Let us assume, for the moment, that (1.8) admits a solution $V_n(S,t)$ that, when regarded as a function of the active variable S, lies in $\mathcal{M}(\eta, \nu) \cap C^2(\mathbb{R})$ for some $\eta < \nu$ satisfying (3.10), and that $\dfrac{\partial V_n}{\partial t}$ lies in $\mathcal{M}(\eta, \nu)$ and satisfies

$$\frac{\partial}{\partial t} M[V_n(\cdot, t)] = M\left[\frac{\partial V_n}{\partial t}(\cdot, t) \right]. \tag{3.15}$$

Let us suppose that $f(S) \in \mathcal{M}(\eta, \nu)$ and let $f^*(z) = M[f(S)]$. By applying the Mellin transform to (1.8) and taking into account properties (3.11) and (3.15), and denoting $v_n(z,t) = M[V_n(\cdot, t)](z)$, one gets

$$\frac{\partial v_n}{\partial t} + (p(z) + Azf_n(t))v_n = 0 \qquad 0 < t < T, \tag{3.16}$$

with the final condition

$$v_n(z, T) = f^*(z), \tag{3.17}$$

where

$$\left. \begin{aligned} p(z) &= p(\alpha + i\omega) = -\frac{1}{2}\sigma^2\omega^2 - i\lambda\omega + q, \\ \lambda &= r - \left(\alpha + \frac{1}{2} \right)\sigma^2, \\ q &= \frac{1}{2}\sigma^2\alpha^2 + \alpha\left(\frac{1}{2}\sigma^2 - r \right) - r. \end{aligned} \right\} \tag{3.18}$$

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The solution of Eq.(3.16) satisfying (3.17) takes the form

$$v_n(z,t) = f^*(z)e^{p(z)(T-t)+AzI_n(t)},$$ (3.19)

where

$$I_n(t) = \int_t^T f_n(\xi)d\xi$$ (3.20)

By (3.12), it follows that $f^*(z) = \mathcal{F}[e^{-\alpha l}f(e^{-l})]$ and, since $e^{-\alpha l}f(e^{-l}) \in L^1$, then $\mathcal{F}[e^{-\alpha l}f(e^{-l})]$ is bounded (see [6],p.241), and $v_n(z,t)$ lies in L^1 with respects to ω. By applying the Mellin inverse transform in (3.19), using (3.13) and denoting

$$g(l) = e^{-\alpha l}f(e^{-l}),$$ (3.21)

it follows that

$$\begin{aligned} V_n(S,t) &= M^{-1}[v_n(z,t)](S) \\ &= S^{-\alpha}\mathcal{F}^{-1}\left[\mathcal{F}[g(l)]e^{p(z)(T-t)+AzI_n(t)}\right](-\ln S). \end{aligned}$$ (3.22)

By (3.6) and (3.7), it is easy to show that

$$\begin{aligned} \gamma_n(x,t) &= \mathcal{F}^{-1}\left[e^{p(z)(T-t)+AzI_n(t)}\right], \\ &= \frac{e^{q(T-t)+\alpha AI_n(t)}}{\sigma\sqrt{2\pi(T-t)}}e^{-\frac{[\lambda(T-t)-x-AI_n(t)]^2}{2\sigma^2(T-t)}} \end{aligned}$$ (3.23)

where q is given by (3.18) and $\gamma_n(\cdot,t)$ lies in L^1 for each $t \in (0,T)$. As g and $\gamma_n(\cdot,t)$ lie in L^1, then $g*\gamma_n(\cdot,t)$ also lies in L^1, and from (3.5) one gets

$$\mathcal{F}[g(l)]e^{p(z)(T-t)+AzI_n(t)} = \mathcal{F}[g*\gamma_n].$$ (3.24)

By (3.22),(3.23)and (3.24) and the inverse Fourier transform theorem, it follows that

$$\begin{aligned} V_n(S,t) &= S^{-\alpha}(g*\gamma_n(\cdot,t))(-\ln S), \\ &= \frac{e^{q(T-t)+\alpha AI_n(t)}}{\sigma\sqrt{2\pi(T-t)}}S^{-\alpha}\int_{-\infty}^{\infty}g(l)e^{-\frac{[\ln S+l+\lambda(T-t)-AI_n(t)]^2}{2\sigma^2(T-t)}}dl. \end{aligned}$$ (3.25)

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3.3 Solution of modified Black-Scholes equation

Since $\left| g(l)e^{-\frac{[\ln S+l+\lambda(T-t)-AI_n(t)]^2}{2\sigma^2(T-t)}} \right| \leqslant |g(l)| \in L^1$, for all $s \in \mathbb{R}$ and $t \in (0,T)$, taking into account (3.14) and the dominate convergence theorem, it follows that $\{V_n(S,t)\}$ is pointwise convergent to the function $V(S,t)$ defined by

$$V(S,t) = \begin{cases} \dfrac{e^{q(T-t)+\alpha A}}{\sigma\sqrt{2\pi(T-t)}} S^{-\alpha} \int_{-\infty}^{\infty} g(l)e^{-\frac{[\ln S+l+\lambda(T-t)-A]^2}{2\sigma^2(T-t)}} \, dl & 0 < t < t_d, \\[3mm] \dfrac{e^{q(T-t)}}{\sigma\sqrt{2\pi(T-t)}} S^{-\alpha} \int_{-\infty}^{\infty} g(l)e^{-\frac{[\ln S+l+\lambda(T-t)]^2}{2\sigma^2(T-t)}} \, dl & t_d < t < T. \end{cases}$$
(3.26)

Taking into account (3.21) and the expressions of λ and q given by (3.18), we get

$$V(S,t) = \begin{cases} \dfrac{e^{q(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} f(e^{-l})e^{-\frac{[\ln S+l+(T-t)(r-\frac{\sigma^2}{2})-A]^2}{2\sigma^2(T-t)}} \, dl & 0 < t < t_d, \\[3mm] \dfrac{e^{q(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} f(e^{-l})e^{-\frac{[\ln S+l+(T-t)(r-\frac{\sigma^2}{2})]^2}{2\sigma^2(T-t)}} \, dl & t_d < t < T. \end{cases}$$
(3.27)

In order to show that (3.27) is a financially admissible solution of problem (1.3) and (1.4), let us introduce the functions

$$\hat{V}_1(S,t) = \hat{V}_2(e^{-A}S,t), \qquad t < T, \tag{3.28}$$

and

$$\hat{V}_2(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} f(e^{-l})e^{-\frac{[\ln S+l+(T-t)(r-\frac{\sigma^2}{2})]^2}{2\sigma^2(T-t)}} \, dl \qquad t < T. \tag{3.29}$$

From (3.27), (3.28) and (3.29),

$$V(S,t) = \begin{cases} \hat{V}_1(S,t) & 0 < t < t_d, \\ \hat{V}_2(S,t) & t_d < t < T, \end{cases} \tag{3.30}$$

and

$$\lim_{t\to t_d^-} V(S,t) = \hat{V}_1(S,t_d) = \hat{V}_2(e^{-A}S,t_d) = \lim_{t\to t_d^+} V(Se^{-A},t).$$

Hence, condition (1.7) is satisfied. Let us define

$$\hat{\gamma}_2(s,t) = \mathcal{F}^{-1}\left[e^{p(z)(T-t)}\right] = \frac{e^{q(T-t)}}{\sigma\sqrt{2\pi(T-t)}}e^{-\frac{[-x+\lambda(T-t)]^2}{2\sigma^2(T-t)}}. \tag{3.31}$$

Taking into account (3.26), (3.27), (3.29) and (3.31), and the substitution $S = e^{-x}$, it follows that

$$
\begin{aligned}
e^{-\alpha x}\hat{V}_2(e^{-x},t) &= \frac{e^{q(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty} g(l)e^{-\frac{[-x+l+\lambda(T-t_1^2}{2\sigma^2(T-t)}}\,dl \\
&= (g * \hat{\gamma}_2(\cdot,t))(x).
\end{aligned}
\tag{3.32}
$$

From (3.4), (3.5), (3.31) and (3.32), one gets

$$e^{-\alpha x}\hat{V}_2(e^{-x},t) = \frac{1}{2}\int_{-\infty}^{\infty}\mathcal{F}[g(l)]e^{p(z)(T-t)}e^{i\omega x}\,d\omega. \tag{3.33}$$

Since $g(l)$ lies in L^1, by ([6],p.246) it shows that $\mathcal{F}[g(l)](\omega)$ is a bounded continuous function, and since $\frac{\partial}{\partial t}e^{p(z)(T-t)}$ exists for every $\omega \in \mathbb{R}$ and $t \in (0,T)$, and

$$\left|\mathcal{F}[g(l)]\frac{\partial}{\partial t}e^{p(z)(T-t)}e^{i\omega x}\right| \leqslant M\left(\frac{1}{2}\sigma^2\omega^2 + |\lambda||\omega| + |q|\right)e^{q(T-t)}e^{-\frac{1}{2}\sigma^2\omega^2(T-t)} \in L^1,$$

for sufficiently large $M, \omega \in \mathbb{R}, t \in (0,T)$, by applying the theorem of derivation of parametric integrals ([8], Th. 14.23) to expression (3.33), it follows that

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(e^{-\alpha x}\hat{V}_2(e^{-x},t)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}[g(l)]\left(\frac{\partial}{\partial t}e^{p(z)(T-t)}\right)e^{i\omega x}\,d\omega, \\
&= \left(g * \mathcal{F}^{-1}\left[\frac{\partial}{\partial t}e^{p(z)(T-t)}\right]\right)(x), \\
&= \left(g * \frac{\partial}{\partial t}\hat{\gamma}_2(\cdot,t)\right)(x).
\end{aligned}
\tag{3.34}
$$

Taking into account the substitution $S = e^{-x}$ in (3.34), one gets

$$\frac{\partial}{\partial t}\hat{V}_2(S,t) = S^{-\alpha}\left(g * \frac{\partial}{\partial t}\hat{\gamma}_2(\cdot,t)\right)(-\ln S), \tag{3.35}$$

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where

$$\frac{\partial}{\partial t}\hat{V}_2(x,t) = \left(-q+\frac{1}{2(T-t)}+\frac{(\lambda(T-t)-x)\lambda}{\sigma^2(T-t)}-\frac{[\lambda(T-t)-x]^2}{2\sigma^2(T-t)^2}\right)\hat{\gamma}_2(x,t). \quad (3.36)$$

In order to compute $\frac{\partial}{\partial S}\hat{V}_2(S,t)$ and $\frac{\partial^2}{\partial S^2}\hat{V}_2(S,t)$, note that, by (3.2), one gets

$$\frac{\partial}{\partial x}(g*\hat{\gamma}_2(\cdot,t)) = g*\frac{\partial}{\partial x}\hat{\gamma}_2(\cdot,t),$$

$$\frac{\partial^2}{\partial x^2}(g*\hat{\gamma}_2(\cdot,t)) = g*\frac{\partial^2}{\partial x^2}\hat{\gamma}_2(\cdot,t).$$

Hence, using (3.32) and the substitution $S = e^{-x}$, it follows that

$$\frac{\partial}{\partial S}\hat{V}_2(S,t) = -S^{-\alpha-1}\left(\alpha(g*\hat{\gamma}_2(\cdot,t))+g*\frac{\partial}{\partial x}\hat{\gamma}_2(\cdot,t)\right); \quad (3.37)$$

and

$$\frac{\partial^2}{\partial S^2}\hat{V}_2(S,t) = S^{-\alpha-2}\left((\alpha+\alpha^2)(g*\hat{\gamma}_2(\cdot,t))+(2\alpha+1)\left(g*\frac{\partial}{\partial x}\hat{\gamma}_2(\cdot,t)\right)+g*\frac{\partial^2}{\partial x^2}\hat{\gamma}_2(\cdot,t)\right), \quad (3.38)$$

where, by (3.31), one gets

$$\frac{\partial}{\partial x}\hat{\gamma}_2(x,t) = \left(\frac{\lambda(T-t)-x}{\sigma^2(T-t)}\right)\hat{\gamma}_2(x,t), \quad (3.39)$$

$$\frac{\partial^2}{\partial x^2}\hat{\gamma}_2(x,t) = \left(\frac{[\lambda(T-t)-x]^2}{\sigma^4(T-t)^2}-\frac{1}{\sigma^2(T-t)}\right)\hat{\gamma}_2(x,t). \quad (3.40)$$

From (3.35)-(3.40), it follows that

$$\frac{\partial\hat{V}_2}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2\hat{V}_2}{\partial S^2}+rS\frac{\partial\hat{V}_2}{\partial S}-r\hat{V}_2 = S^{-\alpha}(g*0) = 0, \quad (3.41)$$

for all $t<T$ and, in particular, for $t_d<t<T$.

For $0<t<t_d$, note that, by (3.28) and (3.41), the substitution $S'=e^{-A}S$ and the chain rule of the differential calculus, one gets

$$\frac{\partial\hat{V}_1}{\partial t}(S,t)+\frac{1}{2}\sigma^2 S^2\frac{\partial^2\hat{V}_1}{\partial S^2}(S,t)+rS\frac{\partial\hat{V}_1}{\partial S}(S,t)-r\hat{V}_1(S,t)$$

$$= \frac{\partial \hat{V}_2}{\partial t}(S',t) + \frac{1}{2}\sigma^2\left(e^A S'\right)^2 e^{-2A}\frac{\partial^2 \hat{V}_2}{\partial S'^2}(S',t) + re^A S' e^{-A}\frac{\partial \hat{V}_2}{\partial S'}(S',t) - r\hat{V}_2(S',t)$$
$$= 0 \tag{3.42}$$

From (3.41) and (3.42), it follows that $V(S,t)$ given by (3.27) satisfies (1.5) for $t \neq t_d$.

Now, it will be shown that $V(S,t)$ satisfies the final condition (1.6). Let $\varphi(x)$ and $\varphi_\tau(x)$ be defined by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \tag{3.43}$$

$$\varphi_\tau(x) = \frac{1}{\tau}\varphi\left(\frac{x}{\tau}\right). \tag{3.44}$$

Hence,

$$\int_{-\infty}^{\infty} \varphi_\tau(x)dx = \int_{-\infty}^{\infty} \varphi(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}dx = 1,$$

and

$$|\varphi(x)| < \frac{C}{(1+|x|)^2}$$

for some sufficiently large C and for all $x \in \mathbb{R}$. Taking into account Theorem 8.15 of ([6],p.235), one gets

$$\lim_{\tau \to 0}(g * \varphi_\tau)(x) = g(x) \qquad a.e., \tag{3.45}$$

and for every x at which g is continuous.

Let us consider the function

$$\phi(x,\tau) = \frac{1}{\tau}\varphi\left(\frac{x}{\tau} - \tau\lambda\right). \tag{3.46}$$

Taking into account (3.43) and the mean value theorem, one gets

$$|\varphi(y) - \varphi(y-h)| \leqslant \max_{y \in \mathbb{R}}|\varphi'(y)||h| = \frac{|h|}{\sigma^2\sqrt{2e\pi}}$$

Hence,

$$|\phi(x,\tau) - \varphi_\tau(x)| = \frac{1}{|\tau|}\varphi\left(\frac{x}{\tau} - \tau\lambda\right) - \varphi\left(\frac{x}{\tau}\right) \leqslant \frac{|\lambda|}{\sigma^2\sqrt{2e\pi}}$$

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and

$$\big| g(x)\big(\phi(x-y,\tau) - \varphi_\tau(x-y)\big)\big| \leqslant \frac{|g(x)||\lambda|}{\sigma^2\sqrt{2e\pi}} \in L^1. \tag{3.47}$$

Furthermore, by applying the L'Hopital rule, it is easy to show that

$$\lim_{\tau\to 0}(\phi(x,\tau) - \varphi_\tau(x)) = 0. \tag{3.48}$$

Taking into account (3.47), (3.48) and the dominated convergence theorem, one gets

$$\lim_{\tau\to 0}(g * (\phi(\cdot,\tau) - \varphi_\tau(\cdot)))(x) = \left(g * \lim_{\tau\to 0}(\phi(\cdot,\tau) - \varphi_\tau(\cdot))\right)(x) = 0. \tag{3.49}$$

Taking $\tau = \sqrt{T-t}$ and using (3.43), (3.44) and (3.46), it follows that expression (3.29) can be written in the form

$$
\begin{aligned}
\hat{V}_2(e^{-x}, t) &= e^{q\tau^2}e^{\alpha x}(g * \phi(\cdot,\tau))(x), \\
&= e^{q\tau^2}e^{\alpha x}\Big(g * \big(\phi(\cdot,\tau) - \varphi_\tau(\cdot)\big)(x) + \big(g * \varphi_\tau(\cdot)\big)(x)\Big)
\end{aligned} \tag{3.50}
$$

Hence, by (3.45) and (3.49), it follows that $\lim_{t\to T^-} \hat{V}_2(e^{-x}, t) = e^{\alpha x}g(x)$, a.e., and therefore

$$\lim_{t\to T^-} \hat{V}_2(S, t) = f(S), \tag{3.51}$$

almost everywhere for S and for every S at which f is continuous.

Chapter 4

Numerical experiments

The integral expression (3.27) can only be computed in analytical form, for a very special payoff function $f(S)$. Thus, it is convenient to apply some numerical technique for computing such integrals.

4.1   Gauss-Hermite scheme

We consider first the Gauss-Hermite approach. Note that, making the substitution

$$u = \frac{\ln S + l + (T - t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{2(T - t)}},$$

in (3.29) and taking into account (3.28) and (3.30), one gets

$$V(S, t) = \begin{cases} \dfrac{e^{-r(T-t)}}{\sqrt{\pi}} I(Se^{-A}, t) & 0 < t < t_d \\ \dfrac{e^{-r(T-t)}}{\sqrt{\pi}} I(S, t) & t_d < t < T, \end{cases} \tag{4.1}$$

where

$$I(S, t) = \int_{-\infty}^{\infty} e^{-u^2} F(u, S, t) du, \tag{4.2}$$

and

$$F(u, S, t) = f\left(e^{-u\sigma\sqrt{2(T-t)} + \ln S + (T-t)(r - \frac{\sigma^2}{2})}\right).$$

We recall that the Gauss-Hermite formula takes the form ([9],p.96),

$$\int_{-\infty}^{\infty} e^{-u^2} F(u)du \approx \sum_{k=1}^{n} \omega_k F(u_k) \tag{4.3}$$

where the nodes u_k are the zeros of the Hermite polynomial

$$H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^2} e^{-u^2}$$

and

$$\omega_k = \frac{2^{n+1} n! \sqrt{\pi}}{\left(H_{n+1}(u_k)\right)^2}.$$

The Gauss-Hermite quadrature formula is very efficient if the integrand is continuous. However, if the integral function presents jumps, the result is not satisfactory because the formula disregards the specific change in the integrand outside the set of zeros of the Hermite polynomial.

4.2   Simpson scheme

This above fact motivates a numerical alternative integration approach that uses the specific information of the integrand close to the parts of the domain with stronger changes. One possibility is to use the composite Simpson's rule after transforming the integration domain into a new finite domain.

Let us consider the substitution $l = \tan u$ into (3.29). Using (3.28) and (3.30), one gets

$$V(S,t) = \begin{cases} \dfrac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}} H(Se^{-A}, t), & 0 < t < t_d \\ \dfrac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}} H(S, t) & t_d < t < T, \end{cases} \tag{4.4}$$

where

$$H(S,t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(e^{-\tan u}) e^{-\frac{\left[\ln S + \tan u + (T-t)(r - \frac{\sigma^2}{2})\right]^2}{2\sigma^2(T-t)}} \sec^2 u\, du$$

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Let us denote

$$G(u, S, t) = f(e^{-\tan u})e^{-\frac{\left[\ln S + \tan u + (T-t)(r - \frac{\sigma^2}{2})\right]^2}{2\sigma^2(T-t)}} \sec^2 u$$

Note that, as $f(S)$ lies in $\mathcal{M}(\eta, \nu)$, it follows that $G(-\frac{\pi}{2}, S, t) = G(\frac{\pi}{2}, S, t) = 0$, for all $0 < t < T, S \in \mathbb{R}$. In this case, the composite Simpson rule takes the form

$$H(S, t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G(u, S, t)du \approx \frac{2h}{3}\sum_{k=1}^{m-1} G(u_{2k}, S, t) + \frac{4h}{3}\sum_{k=1}^{m} G(u_{2k-1}, S, t), \quad (4.5)$$

where $u_k = -\frac{\pi}{2} + hk$ for $k = 1, \ldots, 2m$ and $h = \frac{\pi}{2m}$.

4.3 Example

Consider the valuation problem of binary options (see [2],p.151), with payoff function $f(S) = \beta H(S - E)$, where β is a positive constant and $H(S - E)$ is the Heaviside function.

It is easy to show that the valuation solution of the problem of binary options with discrete dividend is given by

$$V(S, t) = \beta e^{-r(T-t)} \begin{cases} N\left(d_2 - \dfrac{A}{\sigma\sqrt{T-t}}\right), & 0 < t < t_d, \\ N(d_2), & t_d < t < T. \end{cases} \quad (4.6)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\frac{S}{E} + (T-t)\left(r + \frac{\sigma^2}{2}\right)\right],$$

$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\frac{S}{E} + (T-t)\left(r - \frac{\sigma^2}{2}\right)\right],$$

$$d_i' = d_i - \frac{A}{\sigma\sqrt{T-t}} \qquad i = 1, 2$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi,$$

To find out how accurate of two numerical quadrature schemes, we programme in Matlab to implement these two methods. We set in both cases the European call option with the exercise price 95, the interest rate 4%, $\sigma = 0.05$, the maturity $T = 365$, $\beta = 1.2$, the dividend will paid on the 180 days and the constant $A = 5$. The results are illustrated in Figure 1 $\sim$ Figure 4:

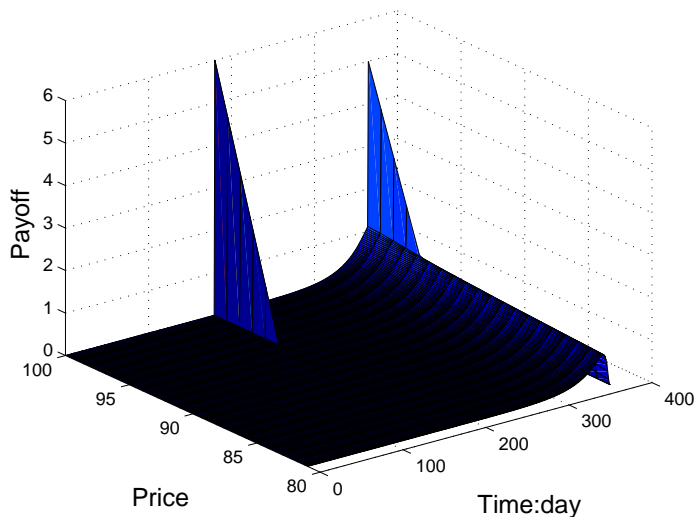By the Matlab error function erf() we calculate the payoff of the option.

Figure 1: Payoff calculated directly from (4.6)

From the Figure 2 and Figure 3, it's obvious that the Gauss-Hermite quadrature formula provides an inaccurate approximation.
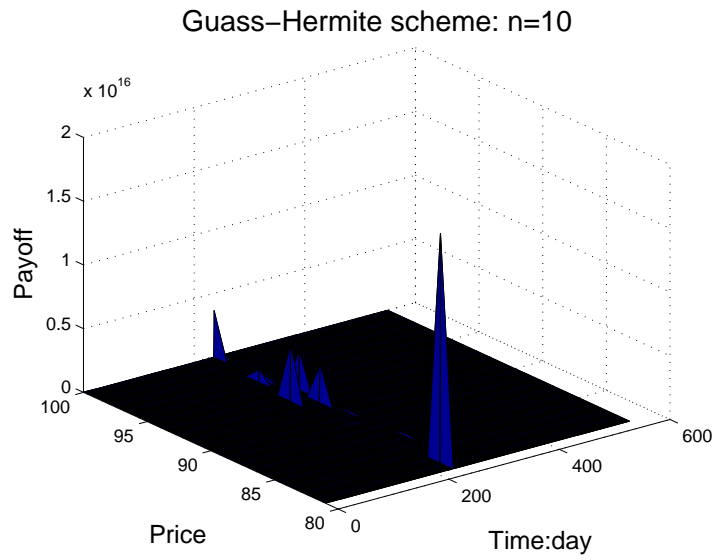
Figure 2: Payoff approximated by Gauss-Hermite polynomial of degree 10
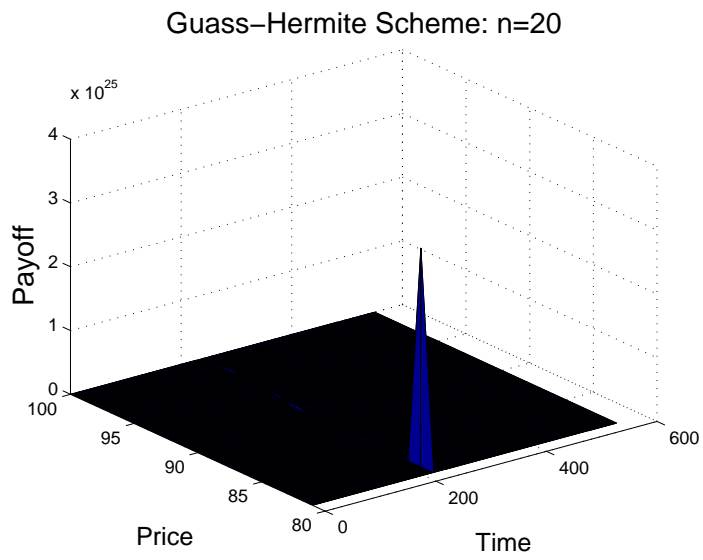

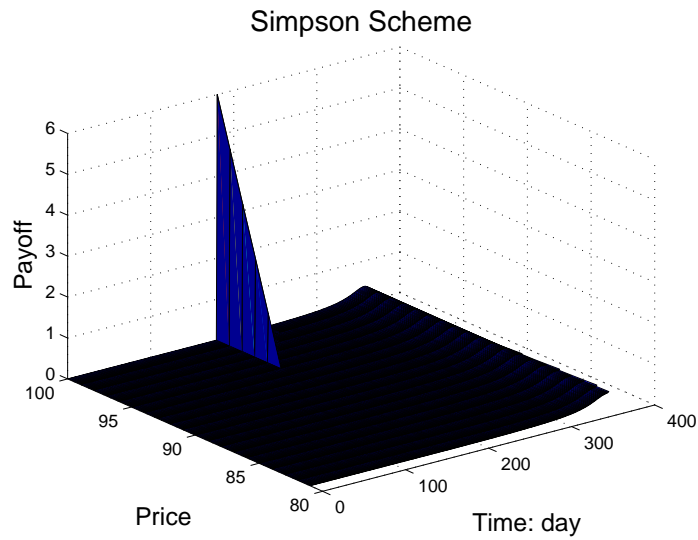Figure 3: Payoff approximated by Gauss-Hermite polynomial of degree 20

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Figure 4: Payoff approximated by Simpson scheme

Compared with Gauss-Hermite scheme, the Simpson scheme gives excellent approximation. From Figure 5, it illustrates that the Simpson scheme provides accurate approximation cross the dividend payment date. However, since we use the same time-step during the final period of the option, the approximation is not accurate when approaching the expiry date.
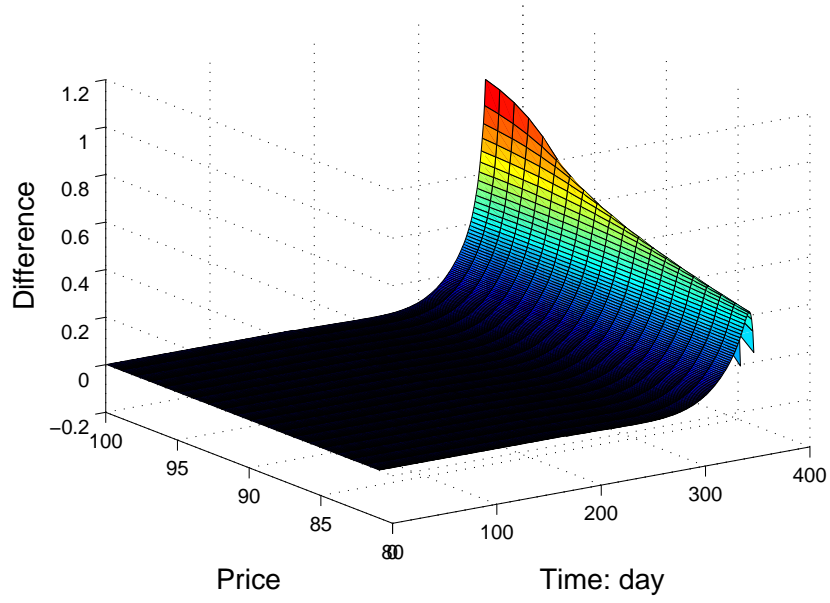
Figure 5: Difference between explicit solution and Simpson approximation

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