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***BMI Paper***

# Bivariate Archimedean copulas:

# an application to two stock market indices

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Abstract

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In order to be able to capture the asymmetry and the non-linearity's in the dependence structure between two vectors, the copula approach was applied. The focused is on the Clayton, the Gumbel and the Frank copula from the Archimedean class. These Archimedean copulas were fitted to a portfolio that consists from two major stock indices from the Eurozone, namely to the German DAX-30 and to the French CAC-40 index. As a result it was found that the copula that best fit the dependence structure between the two indices is characterized by a relatively strong upper tail dependence described by the Gumbel (3.84) copula.

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## Preface

This research paper is written as one of the last mandatory courses of the master study Business Mathematics & Informatics at the Vrije Universiteit (VU). The aim of this paper is to prepare students at the last phase of their study writing their master thesis. In this research paper, the three pillars of the study should be reflected: the mathematics, the business component and the informatics.

I would like to thank Dr. Michael Massmann for helping me writing this paper and for his patience.

Maria Mahfoud

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## Contents

# 1. Introduction and Background

Identifying and quantifying dependencies is the core of econometric modeling, especially in the risk management. Historically measuring and modeling dependencies has centered on the Pearson correlation because of its ease in use. However, the Pearson correlation is not a measure of general but only of linear dependence (Aas, 2006). Embrechts, McNeil & Straumann (2002) argue that the Pearson correlation suffer from several shortcomings when looking at the four desired properties of dependence measures. So the Pearson correlation is not invariant under non-linear transformations, which are frequently used especially when working with financial data. According to Embrechts, McNeil & Straumann (1999) it is better to use rank correlation measures such as Kendall's tau and Spearman's rho instead of Pearson correlation. However, they argued that one should use a dependence measure that reflects more knowledge about the dependence structure instead of summarizing in just a single number. Embrechts et al. (2002) indicates the superiority of copulas in modeling dependence because of their flexibility and the various types of dependence that they allow for.

Copulas were first introduced by Sklar (1959). The benefits from using copulas in finance were indicated in 1987 by Genest and MacKay. The study of Embrechts et al. (1999) was the first study that uses copulas in the financial context.

Estimating the joint distribution of risk factors is in general a hard task. The copula approach provides a way of isolating the marginal behavior from the dependence structure (McNeil, Frey and Embrechts (2005). Alcock and Hatherley (2007) suggested that through copulas, the non-normal dependence structure could be modeled by using only uniforms of the marginal distributions which, in turn, permitted to avoid the restrictive assumptions of normality and linear dependence. This means that the marginals can be modeled using each type of distributions without influencing the dependence structure between them.

Three types of copulas can be distinguished: fundamental copulas, implicit copulas and explicit copulas. Fundamental copulas are copulas that represent perfect positive dependence, independence and perfect negative dependence. Implicit copulas are copulas extracted from well-known multivariate distributions and do not have closed form expressions. Explicit copulas, also called Archimedean copulas, are copulas with simple closed form expressions and follow general mathematical constructions to yield copulas. This research focuses on the last type of copulas, namely the Archimedean copulas. Archimedean copulas represent a class of copulas that are broadly used to model the dependence structure between risk factors.

Archimedean copulas have a simple closed from and do not need to be represented by a multivariate distribution. This class of copulas became very popular due to the easiness of the construction and the implementation of their copulas next to the wide range of dependence that they allow for (Cherubini, Luciano and Vecchiato, 2004).

This research focuses on bivariate Archimedean copulas. The most important copulas within this class will be discussed in details: namely the Clayton copula, the Gumbel copula and the Frank copula.

In this research we will try to answer the following questions:

1) What are Archimedean copulas?
2) How are Archimedean copulas linked to dependence measures?
3) How can Archimedean copula be fitted to financial data?

The remainder of this thesis is organized as follows. Section 2 discusses in details different dependence measures. Section 3 introduces the copula theory and how copulas can be related to dependence measures. Section 4 illustrates an application of the Archimedean to a portfolio composed from two stock indices. Section 4 provides conclusions. Recommendations for further research are discussed in Section 6.

# 2. Dependence measures

In this section the desirable properties of dependence measures are discussed. Furthermore, different types of dependence measures will be discussed namely: the Pearson linear correlation, the rank correlation and the tail dependence. It will be shown that the Pearson linear correlation does not always meet the desired properties of dependence measures. Further, the copula based dependence measures (rank correlation and tail dependence) which meet the desired properties will be discussed. Since this research focuses on the bivariate copulas, the discussion of dependence will be restricted to the bivariate case.

## 2.1 Desirable properties of dependence measures

Before discussing the desirable properties of dependence measures an introduction to the concept of dependence is needed. Two random variables X and Y are said to be dependent or associated if they do not satisfy the independence property:$(X, Y) = F_1(X) * F_2(Y)$, where $F_1(X)$ and $F_2(Y)$ are the marginal distributions functions of the random variables X and Y (Trivedi, 2005).
Let δ express a simple scalar measure of dependence. Four desired properties of δ are described by (Embrechts, McNeil & Straumann, 2002).

(I) $\delta(X, Y) = \delta(Y, X)$, known as the condition of symmetry.

(II) $-1 \leq \delta(X, Y) \leq 1$, known as the condition of normalization.

(III) $\delta(X, Y) = 1$, then $(X, Y)$ are comonotonic.

$\delta(X, Y) = -1$, then $(X, Y)$ are countermonotonic.

(IV) $\delta(T(X), Y) = \begin{cases} \delta(Y, X) T \text{ increasing} \\ -\delta(Y, X) T \text{ decreasing} \end{cases}$

For strictly monotonic transformation $T: \mathbb{R} \to \mathbb{R}$ of X.

## 2.2 Pearson linear correlation

Pearson linear correlation is the most widely used type of dependence measures. The Pearson linear correlation measures the direction and the degree to which one variable is linearly related to the other variable. For non-degenerating random variables X and Y, the linear correlation coefficient is defined by:

$$\rho_{(X,Y)} = \frac{cov[X,Y]}{\sigma_X \sigma_Y}$$

Where $cov[X,Y]$ is the covariance between $X$ and $Y$, $\sigma_X$ and $\sigma_Y$ are the standard deviations of the random variables $X$ and $Y$ respectively. The Pearson linear correlation takes values between -1 and 1. When $\rho_{(X,Y)} = 1$ then the variables X and Y are said to be perfectly dependent by an increasing relationship. When $\rho_{(X,Y)} = -1$ then the variables X and Y are perfectly dependent by a decreasing relationship. Furthermore, if the random variables $X$ and $Y$ are independent then the correlation between these two variables is equal to zero (Embrechts, McNeil and Straumann, 2002).

The popularity of the Pearson linear correlation coefficient is mainly due to its ease in use and intuitive comprehension (Cherubini, Luciano and Vecchiato, 2004). However, when looking back at the four desired properties of dependence measures, proposed by Embrechts, McNeil and Straumann (2002), several shortcomings can be detected. Independency implies that the correlation is equal to zero but zero correlation does not imply that the random variables are independent. Consider two random variables, $X \sim N(0,1)$ and $Y = X^2$**.** In this case, the correlation between $X$ and $Y$ is equal to zero. However, the variables X and Y are perfectly dependent. So a linear correlation coefficient of zero only needs the covariance between X and Y to be equal to zero. Whereas zero dependence requires $cov[\phi_1(X), \phi_2(Y)] = 0$ for each function of $\phi_1$ and $\phi_2$. So if the variables $(X,Y)$ are independent, then the Pearson linear correlation coefficient equals zero. However, the converse is not true because $\rho_{(X,Y)}$ only detects linear dependence. Only in the case where $(X,Y)$ are jointly normally distributed, uncorrelatedness is equivalent to independence (Embrechts, McNeil and Straumann, 1999). Another shortcoming of the Pearson linear correlation is that is not defined when the variance of X or Y is not finite. This means that the linear correlation is not a suitable dependence measure when dealing with distributions that are characterized by fatter tails, which is the case in the most financial time series data (Cherubini, Luciano and Vecchiato, 2004).

Further, the Pearson linear correlation does not satisfy the invariance property (the fourth desirable property of dependence measures). This means that the Pearson linear correlation is not invariant under non-linear monotone transformations. This can be explained by the fact that linear correlation does not only depend on the joint distribution of r.v.s but also on their marginals (Cherubini, Luciano and Vecchiato, 2004).

Anscombe (1973) argued in his paper on the role of the graphs in statistical analysis that the Pearson linear correlation can be misleading if it not combined with scatter plots. He introduced the Anscombe's quartet where he illustrates four datasets with the same statistical properties but the variables show different relationships (Figure A1-Appendix A).

## 2.3 Rank correlation

In order to deal with the shortcoming of the Pearson linear correlation, two rank correlation measures will be discussed namely: the Spearman's rank correlation ($\rho_S$) and the Kendall's rank correlation ($\tau_K$).Because these two measures both lay on the concept of concordance, this concept will be introduced first.

The observations ($X_1,\ Y_1$) and ($X_2, Y_2$)are said to be concordant if ($X_1 < X_2$) and ($Y_1 < Y_2$) or if ($X_1 > X_2$) and ($Y_1 > Y_2$) . This means that large (small) values of the random variable X are associated with large (small) values of the random variable Y. If the opposite is true, discordance arises. In the following the two rank correlation measures will be discussed in more details.

### 2.3.1 Spearman's rank correlation

The spearman linear correlation is a non-parametric correlation measure defined by (Embrechts, McNeil, and Straumann, 2002):

$$\rho_S = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

Where n is the number of the paired ranks, and d is the difference between the paired ranks. In this sense, Spearman's rank correlation can be seen as the Pearson linear correlation between the ranked variables. The variables are ranked by assigning the highest rank to the highest value.

The most attractive property of Spearman's rank correlation is that it does not make any assumption about the frequency distribution of the two variables. Another attractive feature of Spearman's rank correlation is its ability to capture the non-linear dependence between the two variables.

### 2.3.2 Kendall's tau

Kendalls's rank correlation is a non-parametric correlation measure that measures the difference between the probability of concordance and the one of discordance between the r.v.s X and Y (Cherubini, Luciano and Vecchiato, 2004). Except that Kendall's rank correlation represents a probability, it is considered equivalent to Spearman's rank correlation. Kendall's tau is given by (Embrechts, McNeil, and Straumann, 2002):

$$\tau_K = \frac{(C - D)}{n(n-1)/2}$$

Where C is the number of concordant pairs and D is the number of discordant pairs.
Like the Spearman's rho, Kendall's rank correlation is invariant under monotonic non-linear transformations of the underlying variables.

If X and Y are variables with continuous marginal distributions and unique copula then Spearman's rho and Kendall's tau can be expressed as follows (Cherubini, Luciano and Vecchiato, 2004):

$$\rho_S(X,Y) = 12\int_0^1\int_0^1 C(u,v)dudv - 3$$

$$\rho_\tau(X,Y) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1$$

Where $C(u, v)$ is the copula of the bivariate distribution function of X and Y. Both $\rho_S(X, Y)$ and may be considered as measures of the degree of monotonic dependence between X and Y, whereas linear correlation measures the degree of linear dependence only. According to (McNeil, Frey and Embrechts, 2005), it is slightly better to use these measures than the linear correlation coefficient. In their opinion, however, one should choose a model for the dependence structure that reflects more detailed knowledge of the problem at hand instead of summarizing dependence with a single number like linear correlation or rank correlation (Aas, 2004). One such measure is tail dependence which is discussed in the next sub-section.

## 2.4 Tail dependence

Tail dependence is a dependence measure that looks at the concordance between extreme values (tail of the joint distribution) of the r.v.s X and Y. This dependence measure is the most appropriate when interested in the probability that one variable exceeds or falls below some given threshold. Geometrically, tail dependence measures the dependence between X and Y in the upper-right and lower-left quadrant of the joint distribution function (Cherubini, Luciano and Vecchiato, 2004). According to Nelson (2006), the parameter of asymptotic lower tail dependence, noted by $\lambda_L$, is the conditional probability in the limit that one variable takes a very low value, given that the other also takes a very low value. Similarly, the parameter of the asymptotic upper tail dependence $\lambda_U$, is the conditional probability in the limit that one variable takes a very high value, given that the other also takes a very high value. The asymptotic tail dependence parameters for copula function are given by Nelson (2006):

$$\lambda_L = \lim_{t \to 0^+} \frac{C(t,t)}{t}$$

$$\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - C(t,t)}{1 - t}$$

The most attractive property of tail dependence measures is that they are independent of the marginal distributions of the variables and that they are invariant under strictly monotone transformations of X and Y. The variables X and Y are said to be asymptotically independent if $\lambda_U(X,Y) = \lambda_L(X,Y) = 0$ but the controversy is not true.

## 3. Copula's

Copulas are parametrically specified joint distributions generated from given marginals (Trivedi, 2005). Therefore properties of copulas are analogous to properties of joint distributions. The main advantage of copulas is that they enable us to separate the marginal behavior and the dependence structure of the variables from their joint distribution function. This separation explains the modeling flexibility provided by copulas and thus the wide interest in copulas for modeling the dependence structure between variables.

### 3.1 Copula Definition and Properties

A copula is a multivariate distribution function from the unit d-cube $[0, 1]^d$ to the unit interval $[0, 1]$ which satisfies the following properties:

- $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i \ \forall\ i \ \leq d \ and \ \ u_i \epsilon\ [0,1]$
  This property means that if the realizations of the d-1 variables are known with marginal probability one, then the d outcomes of the joint probability is equal to the one with uncertain outcome ($u_i$).
- $C(u_1, \dots, u_d) = 0\ if\ \ u_i$ =0 $\forall\ i \leq d$. This property says that if the realization of one variable has the marginal probability zero than the joint probability of all outcomes is zero. This property is also known as the grounded property.
- $C\ is\ d - increasing$. This property ensures that the joint probability will be not negative. This is because the volume ( C) of any d-dimensional interval is non-negative
- Fréchet Bounds

  $$max\left\{\sum_{i=1}^{d} u_i + 1 - d, 0\right\} \leq C(u) \leq min\{u_1, \dots, u_d\}$$

  The upper bound is called the Frechet –Hoeffding upper bound and the lower bound is called the Frechet –Hoeffding lower bound.

**Sklar's Theorem**

Let F be a joint distribution function with margins $F_1, \dots F_d$.There exist a copula such that for all $x_1, \dots, x_d$ in [−   ,   ]

$$F(x_1, \dots, x_d) = C\big(F_1(x_1), \dots, F_d(x_d)\big)$$

If the margins are continuous then C is unique; otherwise C is uniquely determined on $Ran\ F_1 * Ran\ F_2 \dots * Ran\ F_d$. And conversely, if C is a copula and $F_1, \dots F_d$ are univariate distribution functions, then $F$ defined above is a multivariate distribution function with margins $F_1, \dots F_d$.
This theorem states that not only are copulas joint distribution functions, but also joint distribution functions can be written in terms of a copula and of the marginal distributions. That is the reason why modeling joint distributions can be reduced to modeling copulas (Schweizer, 1991).

Sklar's theorem states also that copulas represents the dependence between the variables that results from splitting the joint distribution into a copula and the marginals. This is why copulas are also called dependence functions (Deheuvels, 1978).

Since this research focus on bivariate copulas some probabilistic properties of copulas will be discussed. If the random variables X and Y are standard uniform distributed:

$$F_1(X) \sim U_1 \ and\ F_2(Y) \sim U_2$$

Then Sklar's theorem involves:

$$Pr(X \le x, Y \le y) = C\big(F_1(x), F_2(y)\big)$$

$$Pr(X \le x, Y > y) = F_1(x) - C\big(F_1(x), F_2(y)\big)$$

$$Pr(X > x, Y \le y) = F_2(y) - C\big(F_1(x), F_2(y)\big)$$

$$Pr(X \le x \mid Y \le y) = \frac{C\big(F_1(x), F_2(y)\big)}{F_2(y)}$$

$$Pr(X \le x \mid Y > y) = \frac{F_1(x) - C\big(F_1(x), F_2(y)\big)}{1 - F_2(y)}$$

$$Pr(X \le x \mid Y = y) = C_{1|2}\big(F_1(x), F_2(y)\big) = \frac{\partial C(v, z)}{\partial z} | v = F_1(x), z = F_2(y)$$

As copulas are dependence measures, they allow us to distinguish the perfect dependence and the independence in a straightforward way.

*Independence:*

The random variables X and Y are says to be independent if C they have the product copula $C^{\perp}$, i.e., $C\big(F_1(x), F_2(y)\big) = F_1(x)F_2(y)$.

*Comonotonicity:*

The perfectly positive dependence or comonotonicity is equivalent to the Fréchet-Hoeffding upper bound. For any $(x_1, y_1), (x_2, y_2)$ a comonotonic set is that for which:

$$\begin{cases} x_1 \leq y_1 \\ x_2 \leq y_2 \end{cases} or \begin{cases} x_1 \geq y_1 \\ x_2 \geq y_2 \end{cases}$$

This property says that higher (lower) realizations of the variable X correspond with higher (lower) realizations of Y.

*Countermonotonicity:*

Also called perfect negative dependence, obtained when the copula attains the Fréchet-Hoeffding lower bound.

For any $(x_1, y_1), (x_2, y_2)$ a comonotonic set is that for which:

$$\begin{cases} x_1 \leq y_1 \\ x_2 \geq y_2 \end{cases} or \begin{cases} x_1 \geq y_1 \\ x_2 \leq y_2 \end{cases}$$

Countermonotonicity states that higher (lower) realizations of X correspond with lower (higher) realizations of Y.

*Invariance property:*

The invariance property can be seen as the most attractive property of copulas. This property says that the dependence captured by a copula is invariant under monotone transformations of the marginal distributions. For example, when applying the logarithmic transformation to the marginal distribution the copula will be not affected. The fact that copulas are invariant under decreasing or increasing transformations makes from copulas a powerful tool in applied work.

There are two parametric families of copulas namely implicit copulas and explicit copulas. Implicit copulas do not have a simple closed form. Copulas from this family are implied by well-known multivariate distribution functions. The most known copulas from this class of copulas are the Gaussian copula and the Student's t-copula. Explicit copulas also called Archimedian copulas represent a class of copulas that are broadly used to model the dependence structure in the data. This class of copulas became very popular due to the easiness of the construction and the implementation of their copulas (simple closed form) next to the wide range of dependence that they allow for. Since this research focuses on bivariate Archimedean copulas, the most important copulas within this class will be discussed in details: namely the Clayton copula, the Gumbel copula and the Frank copula. First the general properties of this class of copulas will be discussed, and then each of the early named copulas will be defined. Furthermore, the main properties of each copula will be discussed and how these copulas are related to dependence measures.

## 3.2 Archimedean Copulas

Archimedean copulas are all constructed by specifying a particular generator[1] function $G$, such that (Cherubini, Luciano and Vecchiato, 2004):

Let $\varphi$ be a strict[2] generator, with $(\varphi)^{-1}$ completely monotonic on $[0,\infty)$[0, ∞), than bi-variate Archimedean copula can be defined as:

$$C\ (u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2))$$

An important source of generators for Archimedean copulas consist of the inverses of the Laplace transform of c.d.f. (Feller, 1971)[3].

One of the attractive features of Archimedean copulas is that they are easily related to dependence measures. Genest and MacKay (1986) proved that the relation between the copula generator function and Kendall's $\tau$ in the bivariate case can be given by (Cherubini, Luciano and Vecchiato, 2004):

$$\tau_K = 1 + 4\int_0^1 \frac{\phi(v)}{\phi'(v)}\,dv$$

[1] A generator $\phi$ is a function defined from I to $R^{*+}$ and have the following properties: continuous, decreasing, convex and $\phi(1)$=0.

[2] A strict generator is a generator with $\phi(0)$=+∞.

[3]A function $\varphi$ on (0;∞) is the Laplace transform of a c.d.f. F if and only if $\varphi$ is compelterly monotonic and $\varphi(0) = 1$.

The relation between Archimedean copulas and tail dependency was demonstrated by Joe (1997). The theorem says that if $\varphi$ is a strict generator. If $\varphi'(0)$ is finite and different from zero, then (Cherubini, Luciano and Vecchiato, 2004);

$$C(u_1, u_2) = \varphi^{-1}\big(\phi(u_1) + \phi(u_2)\big)$$

Does not have tail dependence. If the copula has upper tail dependence, then $\frac{1}{\varphi'(0)} = -\infty$

And the coefficients of upper tail dependence and lower tail dependence are given by (Cherubini, 2004):

$$\lambda_U = 2 - 2\ \lim_{s\to 0+} \frac{\varphi'(s)}{\varphi'(2s)} \quad \text{and} \quad \lambda_L = 2\ \lim_{s\to +\infty} \frac{\varphi'(s)}{\varphi'(2s)}$$

In the following part the closed formula for the Frank, Clayton and Gumbel copulas is given. Also the explicit formulas between the copula parameter and the dependence measures will be given.

### 3.2.1 Clayton copula

The Clayton copula is first introduced by Clayton (1978). The Clayton copula is mostly used to study correlated risks because of their ability to capture lower tail dependence. The closed from of the bivariate Clayton copula is given by:

$$C^{Cl}(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

Where $\theta$ is the copula parameter restricted on the interval $(0, \infty)$. If $\theta = 0$ then the marginal distributions become independent; when $\theta \to \infty$ the Clayton copula approximates the Fréchet-Hoeffding upper bound.

Due to the restriction on the dependence parameter, the Fréchet-Hoeffding lower bound cannot be reached by the Clayton copula. This suggests that the Clayton copula cannot account for negative dependence.

The dependence between de Clayton copula parameter and Kendall's tau rank measure is simply given by:

$$\tau_K = \frac{\theta}{\theta + 2}$$

The parameter of lower tail dependence for this copula can be calculated by: $\lambda_L = 2^{-1/\theta}$. (Cherubini, Luciano and Vecchiato, 2004).

While the relation between the Clayton copula parameter and the dependence measures: Kendall's tau and the tail dependence are simple; the association between the copula parameter and the Spearman's rho is very complicated.

### 3.2.2 Gumbel copula

The Gumbel copula (1960) is used to model asymmetric dependence in the data. This copula is famous for its ability to capture strong upper tail dependence and weak lower tail dependence. If outcomes are expected to be strongly correlated at high values but less correlated at low values, then the Gumbel copula is an appropriate choice. The bivariate Gumbel copula is given by:

$$C^{Gu}(u_1, u_2; \theta) = exp\left(-\left[(-\log u_1)^\theta + (-\log u_2)^\theta\right]^{1/\theta}\right)$$

Where $\theta$ is the copula parameter restricted on the interval $[1, \infty)$. When $\theta$ approaches 1, the marginals become independent and when $\theta$ goes to infinity the Gumbel copula approaches the Fréchet-Hoeffding upper bound. Similar to the Clayton copula, the Gumbel copula represents only the case of independence and positive dependence.

The relation between the Gumbel copula parameter and the Kendall's tau is given by:

$$\tau_K = 1 - \theta^{-1}$$

The parameter of the upper and lower tail dependence of the Gumbel copula can be calculated respectively by $\lambda_U = 2 - 2^{1/\theta}$ $and$ $\lambda_L = 0$. While the relation between the dependence measures: Kendall's tau and the tail dependence and the copula parameter have simple closed forms, the relation between the copula parameter and the Spearman's rho has no closed form.

### 3.2.3 Frank copula

The Frank copula (1979) is given by:

$$C^{Fr}(u_1, u_2; \theta) = -\theta^{-1} \log\left\{1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{(e^{-\theta} - 1)}\right\}$$

Where $\theta$ is the copula parameter that may take any real value. Unlike the Clayton and the Gumbel copula, the Frank copula allows the maximum range of dependence. This means that the dependence parameter of the Frank copula permits the approximation of the upper and the lower Fréchet-Hoeffding bounds and thus the Frank copula permits modeling positive as negative dependence in the data. When $\theta$ approaches $+\infty\ and -\infty$ the Fréchet-Hoeffding upper and lower bound will be attained. The independence case will be attained when $\theta$ approaches zero. However, the Frank copula has neither lower nor upper tail dependence ( $\lambda_U = \lambda_L = 0$). The Frank copula is thus suitable for modeling data characterized by weak tail dependence.

The calculation of the Kendall's tau and the Spearman's rho ($\rho_S$) from the copula parameter requires the computation of the Debye[4] function and are given by:

$$\tau_K = 1 + 4[D_1(\theta) - 1]/\theta$$

and

$$\rho_S = 1 - 12[D_2(-\theta) - D_1(-\theta)]/\theta$$

The relationship between the discussed Archimedean copulas and the copula based dependence measures: Kendall's tau, Spearman's rho and tail dependence is summarized in Table 3.1.

| Copula | Kendall's tau | Spearman's rho | Upper tail | Lower tail |
|---|---|---|---|---|
| Clayton | $\frac{\theta}{\theta+2}$ | complicated | 0 | $2^{-1/\theta}$ |
| Gumbel | $1-\theta^{-1}$ | No closed form | $2-2^{1/\theta}$ | 0 |
| Frank | $1+4[D_1(\theta)-1]/\theta$ | $\rho_S = 1-12[D_2(-\theta)-D_1(-\theta)]/\theta$ | 0 | 0 |

Table 3.1:Association between some Archimedean copulas and the rank correlation measures: Kendall and Spearman and the tail dependence

[4] $D_k(\alpha) = \frac{k}{\alpha^k}\int_0^\alpha \frac{t^k}{exp(t)-1}dt,\ k = 1,2$

## 3.3 Copula parameter estimation

All Archimedean copulas that are discussed are characterized by one dependence parameter that needs to be estimated. The most widely used estimation methods are the full maximum likelihood (FML) estimation method and the inference for margins (IFM) approach. In the following sub-sections, these two estimation methods will be discussed in more details.

### 3.3.1 Full maximum likelihood estimation

The FML is the most direct estimation method. Using the FML approach, the copula parameter and the marginal distribution parameters are estimated simultaneously.
Following Sklar's theorem the following canonical form for the bivariate joint density of two random variables $X_1$and $X_2$ can be used (Cherubini, Luciano and Vecchiato, 2004):

$f(x_1, x_2) = c\big(F_1(x_1), F_2(x_2)\big)f_1(x_1)f_2(x_2)$

Where $c\big(F_1(x_1), F_2(x_2)\big) = \frac{\partial^2 C(F_1(x_1),F_2(x_2))}{\partial F_1(x_1)\partial F_2(x_2)}$, $c$ is the copula density, $f_i$ and $F_i$ are the marginal density and distribution function respectively.
Now the log-likelihood function for the bivariate case can be expressed as follows (Cherubini, Luciano and Vecchiato, 2004):

$$l(\theta) = \sum_{t=1}^{T} \log c\left(F_1(x_{1,t}), F_2(x_{2,t})\right) + \sum_{t=1}^{T}\sum_{i=1}^{2} \log f_i(x_{i,t})$$

Then the maximum likelihood estimator maximizes the log likelihood and is given by:

$$\hat{\theta} = \arg\max_{\theta} l(\theta)$$

The estimates of the maximum likelihood parameter can be found using a numerical maximization method. Furthermore under certain regularity conditions the asymptotic theory can be used for both the marginal as the copula. Under these regularity conditions, the maximum likelihood estimator exists, is consistent, asymptotically efficient and asymptotically normal. Furthermore (Cherubini, Luciano and Vecchiato, 2004):

$$\sqrt{T}(\hat{\theta} - \theta_0) \sim N(0, \mathcal{J}^{-1}(\theta_0))$$

Where $\mathcal{J}$ is the Fisher's information matrix and $\theta_0$is the true value.

### 3.3.2 Inference for the margins

Compared to FML, the IFM method is computationally more attractive (Kole et all, 2005).The IFM estimates the marginal distribution parameters separately from the copula parameters. Hence, here the estimation procedure is divided in two steps (Cherubini, Luciano and Vecchiato, 2004).

In the first step the parameters for the models of the marginals are estimated:

$$\hat{\theta}_1 = \arg\max_{\theta_1} \sum_{t=1}^{T} \sum_{i=1}^{2} \log f_i(x_{i,t}; \theta_1)$$

In the second step the parameter of the copula model are estimated, given $\hat{\theta}_1$:

$$\hat{\theta}_2 = \arg\max_{\theta_2} \sum_{t=1}^{T} \log c\big(F_1(x_{1,t}), F_2(x_{2,t}); \theta_2, \hat{\theta}_1\big)$$

This results in the IFM estimator:

$$\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$$

Under certain regularity conditions[5], the IFM estimator verifies the property of asymptotic normality and can be seen as a highly efficient[6] estimator compared to the FML estimator (Joe, 1997).

## 3.4 Copula selection

Up to now, there is no consensus about a statistic criterion that selects the copula that provides the best fit to the data. Dias and Embrechts (2004) and Palaro and Hotta (2006) used the AIC criterion to select the copula that provides the best fit. However, various simulation studies

[5] Regularity conditions include interchange of differentiation, integration and summation. For more details see Joe (1997)
[6] With efficient it is mean asymptotic relative efficient.

shows that the Schwarz Information Criterion (SIC or BIC) performs better in large samples whereas the AIC tends to be superior in small samples (Shumway and Stoffer, 2011). In this research, both criterions were implemented and the copula that provides the best fit is the one that correspond with the lowest values of these criterions. The AIC and the SIC can be defined as follows:

$$AIC = -2 * \log likelihood + 2k$$

$$SIC = -2 * \log likelihood + \ln(n) * k$$

where k is the number of parameters of the copula model and n is the number of observations.

## 4. Application

In this section an application to a portfolio composed by two stock indices will be discussed.

### 4.1 Data description

A portfolio is considered composed by the Germany Dax-30 index and the French CAC-40 index. As these represent the two strongest economies in the Eurozone. The daily closing prices in Euro from DataStream for the period ranging from 03-01-2000 to 22-05-2012 are used. Further, the price indices are transformed to log-returns. This results in 3231 observations per stock index.

Figure 4.1 presents the time plots and the distribution plots of the Dax-30 and the CAC-40 stock indices. Both time plots show the stylized fact of volatility clustering where large (small) returns are followed by large (small) returns. The same time plots show also the effect of the recent financial crisis characterized by high deviations in the return series. Further, Figure 4.1 shows that the distribution plots (mentioned by the straight line) of the Dax-30 and the CAC-40 stock indices deviate from the normal distribution (mentioned by the dotted line). The distribution plots of both return series tend to be more leptokurtic. This can be concluded from the high peak around the mean.

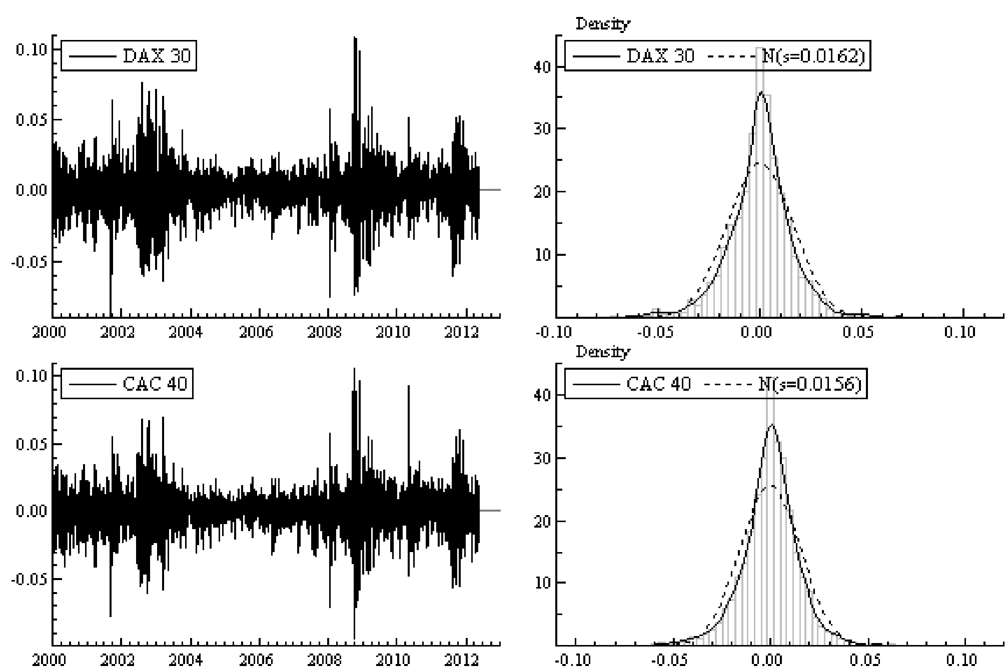


Figure 4.1: Time plots and distribution plots of the DAX-30 and the CAC-40 log-return series

Table 4.1 shows that both return series have a negative mean. Further, both the DAX-30 and the CAC-40 returns have a distribution that is slightly right-tailed (positive skewness). As it was expected from the distribution plots, both return series have a kurtosis value higher than the kurtosis value of the normal distribution (3). This means that both distributions are leptokurtic.

| Stock Index | Mean | Median | Maximum | Minimum | Std. Deviation | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| DAX_30 | -0.00001 | 0.00031 | 0.10798 | -0.08875 | 0.01622 | 0.00774 | 7.18579 |
| CAC_40 | -0.00020 | 0.00000 | 0.10595 | -0.09472 | 0.01565 | 0.02982 | 7.70429 |

Table 4.1: Descriptive statistics of the DAX-30 and the CAC-40 stock indices

The non-normality of the DAX-30 and the CAC-40 returns was assessed by mean of the Jarque-Bera (JB) test. JB test strongly reject the null hypothesis that the return series are normally distributed at the 1% significance level.

The volatility clustering observed from the time plots of the DAX 30 and the CAC-40 stock indices was also tested by mean of Ljung-Box Q-test (LB). The LB test performed on the first 8 lags of the squared log-returns strongly rejects the hypothesis of no serial correlation at the 1% significance level (Table 4.2).

| Stock Index | Jarque-Bera | JB P-value | Ljung Box for squared log returns | LB P-value |
|---|---|---|---|---|
| DAX_30 | 2359 | 0 | 1330 | 0 |
| CAC_40 | 2980 | 0 | 1260 | 0 |

Table 4.2: Jarque-Bera and Liung Box tests to test respectively for the normality of the log returns and for the ARCH effect in the squared log returns

The fact that log return series are not normally distributed and exhibit serial correlation suggest the use of the class of ARMA-GARCH. The ARMA-GARCH model work as a filter which provides serially independent innovations.

## 4.2 Model

To specify the bivariate model, the two models must be specified for the marginal distributions and one model that describes the dependence structure between the marginals.

### 4.2.1 Modeling the marginal distributions

For the reasons discussed early, the univariate log return series will be modeled by the ARMA-GARCH model with student-t distributed error terms. If $X_{i,t}$ is the log return of index i at time t, then the ARMA(p,q)-GARCH(m,s) model is given by:

$$X_{i,t} = \mu_i + \sum_{j=1}^{p} \varphi_{i,j} X_{i,t-j} + \sum_{k=1}^{q} \theta_{i,k} \varepsilon_{i,t-k} + \varepsilon_{i,t}$$

$$\varepsilon_{i,t} = \sigma_{i,t} \eta_{i,t}$$

$$\sigma_{i,t}^2 = \alpha_{i,0} + \sum_{j=1}^{m} \alpha_{i,j}\, \varepsilon_{i,t-j}^2 + \sum_{k=1}^{s} \beta_{i,k} \sigma_{i,t-k}^2$$

Where $i = \{\mathrm{Dax} - 30, \mathrm{CAC} - 40\}$ and

$$\alpha_{i,0} > 0, \alpha_{i,j} \geq 0 \; and \; \sum_{j=1}^{\max(m+s)} \left(\alpha_{i,j} + \beta_{i,j}\right) < 1$$

The parameters of the ARMA-GARCH models are estimated using the conditional likelihood approach, where the log-likelihood function is given by:

$$l(\theta_i) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n} \log(\sigma_{i,t}^2) - \frac{1}{2}\sum_{t=1}^{n} \frac{\varepsilon_{i,t}^2}{\sigma_{i,t}^2}$$

Where $\theta_i = \mu_i, \varphi_{i,1}, \dots, \varphi_{i,p}, \theta_{i,1}, \dots, \theta_{i,q,} \alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,m}, \beta_{i,1}, \dots, \beta_{i,s}$

Then the maximum likelihood estimator maximizes the log likelihood and is given by:

$$\hat{\theta}_i = \arg\max_{\theta_i} l(\theta_i)$$

The maximum likelihood estimator can be found using a numerical maximization approach. Once the parameter estimates are obtained, the model can be used to obtain the one-step ahead forecasts of the volatility and the value of the log-likelihood. For the starting value of $\sigma_{i,0}^2$ the empirical variance of the market index series over the sample period is used.

The identification of the ARMA lags was done empirically by means of PACF and ACF plots for AR lags and MA lags respectively for the return series. The same functions were used for identifying the lag parameters of the GARCH model but for squared log-returns (See Figure BI). Next, the model with the specified lags was estimated. Then a general to specific approach was applied. The estimates were tested for significance by means of a t-test and the parameters that seemed to be not significant at the 1% significance level will be removed. The adequacy of the final model was graphically assessed by means of ACF plots and tested by means of LB test for the presence of serial correlation in the mean or in the variance equation.

The optimal model that seems to fit the DAX-30 returns the best is the ARMA(2,2)-GARCH(1,1) model. The model that provides the best fit to the CAC-40 return series is the ARMA(1,1)-GARCH(1,1) model. The estimated parameters and their standard errors are reported in Table 4.3 (more estimation details can be found in Table BII).

| | DAX-30 | | CAC-40 | |
|---|---|---|---|---|
| Parameter | Estimate | Error | Estimate | Error |
| $\mu$ | 0.001 | 0.000 | 0.001 | 0.000 |
| $\varphi_1$ | 0.427 | 0.050 | 0.613 | 0.113 |
| $\varphi_2$ | -0.905 | 0.033 | - | - |
| $\theta_1$ | -0.436 | 0.042 | -0.672 | 0.117 |
| $\theta_2$ | 0.920 | 0.035 | - | - |
| $\alpha_0$ | 0.000 | 0.006 | 0.000 | 0.005 |
| $\alpha_1$ | 0.090 | 0.011 | 0.087 | 0.011 |
| $\beta_1$ | 0.904 | 0.010 | 0.908 | 0.011 |
| $\upsilon$[7] | 10.401 | 2.171 | 10.744 | 2.048 |

Table 4.3: ARMA-GARCH-t estimation

Table 4.3 shows that the sum of the estimated parameters $\alpha_1$ and $\beta_1$ for both stock indices is lower than 1. This means that the unconditional variance of the error terms is finite where the conditional variance evolves over time (Tsay, 2005).

The adequacy of the estimated models was assessed graphically by drawing ACF plots of the standardized residuals and the squared standardized residuals. Figure BII indicates no significant serial correlation in the standardized residuals and the squared standardized residuals. The adequacy of the estimated models was also tested by mean of LB test. LB test is applied to the standardized residuals and the squared standardized residuals. The LB test fails to reject the null hypothesis of no serial at the first 8[8] lags of the standardized residuals for both stock indices at the 1% significance level. The corresponding p-values are 0.013 and 0.023 for the DAX-30 and

[7] The number of the degrees of freedom corresponding to the estimated student-t distribution

[8] According to Tsay (2005), using LB test with $m \approx \ln(T)$ lags where T is number of observations provides better power performance.

the CAC-40 index, respectively. LB test also fail to reject the null hypothesis for the squared standardized residuals at the first 8 lags for both indices at the 1% significance level. The LB test on the squared standardized residuals corresponding to the DAX-30 shows that LB (8) = 16.206 with a p-value = 0.013. This means that the volatility equation is adequate at the 1% significance level. For the CAC-40 index, the LB test also fails to reject the null hypothesis of no serial correlation in the first 8 lags at the 1% significance level. The LB test has a value equal to 14.665 with a p-value equal to 0.023. From these results it can be concluded that the estimated models are adequate in the sense that they provide serially independent innovations. These serially independent innovations will be used to model the dependence structure between the marginal distributions of the DAX-30 and the CAC-40 stock indices.

### 4.2.2 Modeling the dependence structure

After filtering the data using ARMA-GARCH models, the obtained pair of innovations (standardized residuals) was transformed to uniforms using the estimated student-t distributions. The uniform series will be used as input for the bivariate Clayton, Gumbel and Frank copula discussed in section 3.2.

The estimated parameter corresponding to each copula, the confidence interval (CI) of the estimated parameters the AIC, and the SIC values are reported in Table 4.4.

| Copula | Estimated Parameter | 95% CI | AIC | SIC |
|---|---|---|---|---|
| Clayton | 3.718 | [3.611, 3.825] | -4591.805 | -4585.725 |
| Gumbel | 3.840 | [3.746, 3.935] | -5445.510 | -5439.430 |
| Frank | 13.770 | [13.370, 14.171] | -5301.488 | -5295.407 |

Table 4.4: Copula parameter estimation

Table 4.4 shows that the Gumbel copula is the one that provides the best fit to the data since it has the lowest values for both criterions: the AIC and the SIC.

As discussed under theoretical framework, the Gumbel copula is known as an extreme copula and is suitable for modeling the upper tail dependence. However, the level of dependence between the marginals depends on the value of the copula parameter. As discussed in the theory, when the copula parameter value is equal to 1 then the marginals are independent, and when the parameter value goes to infinity then the Gumbel copula approaches the Fréchet-Hoeffding upper bound and the marginals become comonotonic or perfectly dependent. From the estimated value of the Gumbel copula parameter, a positive dependence is expected between the data. To get more insight in the dependence structure given by this copula, plots of the copula, its density, scatter plot of random sample simulated from this copula and the copula contour plots were drawn (Figure 4.2). As expected from the estimated copula parameter, the plots of the copula and its density illustrated in Figure 4.2 shows that the estimated Gumbel copula is characterized by strong upper tail dependence. The same result can be concluded from the scatter plot where the simulated values are more concentrated in the upper tail.

For the interpretation of the contour plot, this plot needs to be compared with the contours in the case of independence and perfect dependence. Comparing the contours of the estimated Gumbel copula with the contours that correspond to the independence case and the comonotonic case illustrated respectively in Figure CI and Figure CII in Appendix B, it can be seen that the contour plot of the estimated Gumbel copula have more resemblance with the one that illustrate the comonotonic case.

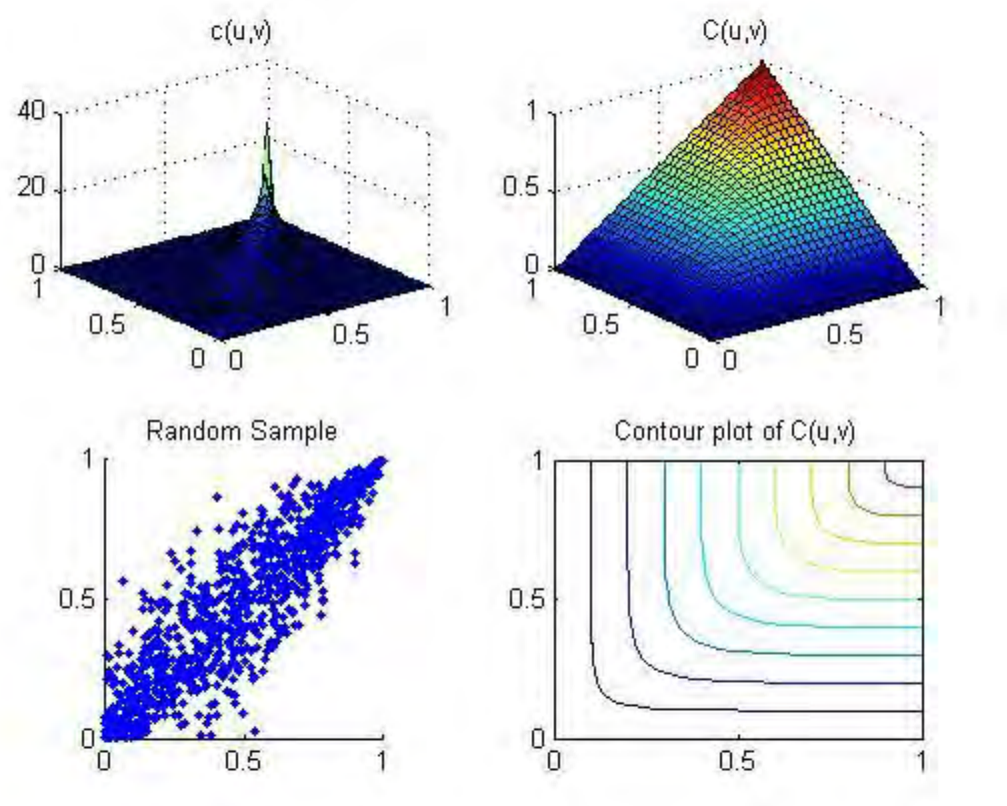


Figure 4.2: Copula density plot (upper left), copula plot (upper right), 1000 simulated points and the copula contour plot of the Gumbel copula with parameter 3.840.

After discussing the dependence generated by the optimal copula, it is interesting to see how the dependence captured by this copula causes the joint distribution to have upper tail dependence. Figure 4.3 shows the density and the contour plot of the distribution obtained by coupling the estimated Gumbel copula and the student-t margins (obtained from the GARCH estimation). This figure shows clear upper tail dependence. This indicates that large gains from the DAX-30 index and the CAC-40 index have more tendencies to occur simultaneously than large losses.

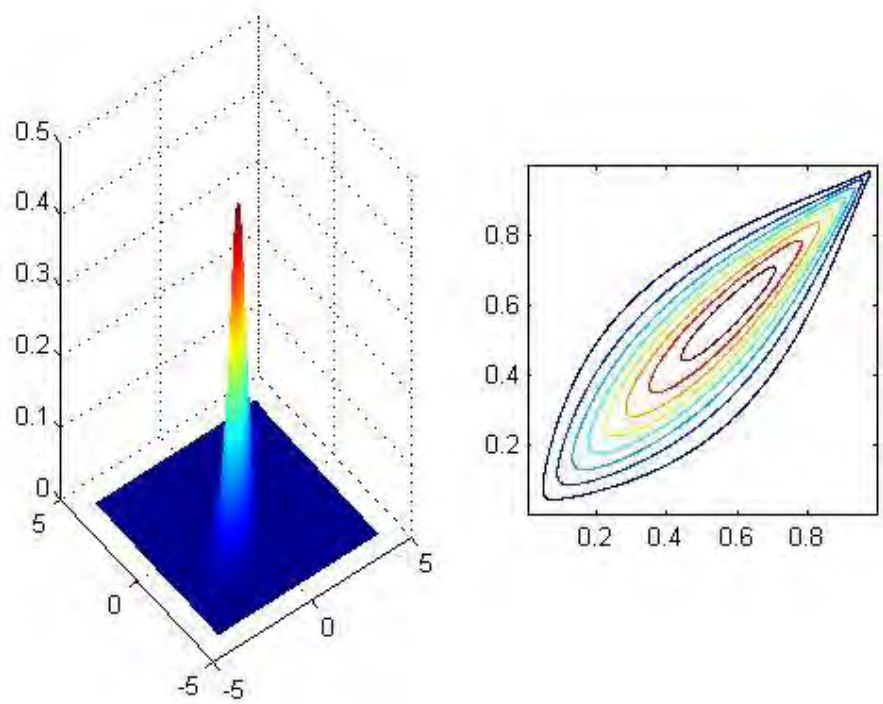


Figure 4.3: Density and contour plot of the joint distribution obtained by coupling the Gumbel (3.84) with the estimated student-t margins

#### *4.2.2.3 Dependence Measures*

Based on the estimated copula and according to the theory described in sub-section 3.2.2, the calculation of the copula based dependence measures: Kendall's tau, Spearman's rho, tail dependence is straightforward. The estimated Kendall's tau, Spearman's rho and tail dependence are illustrated in Table 4.5. As can be seen from Table 4.5, both rank correlation measures: Kendall's tau and Spearman's rho are positive and have a value that is closer to 1. This means that the two stock indices are strongly positively correlated. The estimated Kendall's tau and Spearman's rho are closer to their sample counterparties, which are respectively equal to 0.725

and 0.891. The fact that estimated Kendall's tau and Spearman's rho are close to the Sample ones lead to more flexibility when modeling the dependence with copulas. If one expects that the dependence between risk factors will be captured by a certain Archimedean copula, than computing the sample Kendall's tau or Spearman's rho allow the computation of any joint probability between these risk factors.

Table 4.5 shows that the stock indices exhibit higher upper tail dependence. However, the value of the lower tail dependence is equal to zero. This is because the Gumbel copula is unable to capture the lower tail dependence.

| Kendall's tau | Spearman's rho | Upper tail | Lower tail |
|---|---|---|---|
| 0.740 | 0.905 | 0.802 | 0.000 |

Table 4.5: The Estimated Kendall's tau, Spearman's rho and tail dependence based on the estimated copula

## 5. Conclusion

This paper discussed the shortcomings of the Pearson correlation regarding the desired properties of dependence measures. Among them are: 1) the Pearson correlation is unable to capture the non-linear dependencies. 2) The Pearson correlation is undefined when extreme events are frequently observed. 3) Pearson correlation is not invariant under strictly increasing transformations.
Further the copula approach is discussed, which provide a powerful modeling tool that deals with the shortcomings of the Pearson correlation. Among the different classes of copulas, this paper focused on the Archimedean class due to their simple closed form and the various types of dependence that they allow for. Specifically, this paper discussed one parameter Archimedean copulas: the Clayton, the Gumbel and the Frank copula. Furthermore, the relations between these three copulas and the dependence measures were discussed.

Finally an application of the Archimedean copulas to two stock indices (the DAX-30 and the CAC-40) was considered. To deal with non-normality, heavy tails and heteroskedasticity in the return series an ARMA-GARCH with student-t distributed error terms was applied. The ARMA-GARCH model serves as a filter for the data and provided serially independent innovations. The standardized residuals (serially independent innovations) were transformed to uniform series using their empirical distribution and the estimated student-t distribution from fitting ARMA-GARCH. The discussed Archimedean copulas were fitted to uniform data and the copula that provides the best fit was selected.
Results of this analysis show that the Gumbel copula with a parameter value equal to 3.84 is the one that provides the best fit to our data. Further, the dependence given by this copula was deeply discussed. It was shown also how the given copula causes the joint distribution of the return series to show the same dependence pattern. Results of this analysis show that the joint distribution of the DAX-30 and the CAC-40 stock indices exhibit higher upper tail dependence. This means that large gains from the DAX-30 index and the CAC-40 index have more tendency to occur simultaneously than large losses.
Finally, it was discussed how the rank correlations: Kendall's tau and Spearman's rho, and the tail dependence could be calculated from the estimated copula. The estimated rank correlations

values were also compared to sample ones. It was found the estimated rank correlations are closed to their sample counterparties.

## 6. Suggestions for further research

In this paper a dependence structure of a portfolio that consists of two stock indices was modeled using bivariate Archimedean copulas, however the most portfolios consists in general of more than two indices. In order to model the dependence structure of such portfolios this analysis needs to be extended to the multivariate case.

The only copula that was used that accounts for both tail dependencies is the Frank copula. However, this copula is characterized by weak tail dependency. For further research the use of copulas that allow modeling different tail dependencies such as the Symmetrized Joe Clayton copula is suggested.

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## Appendix A

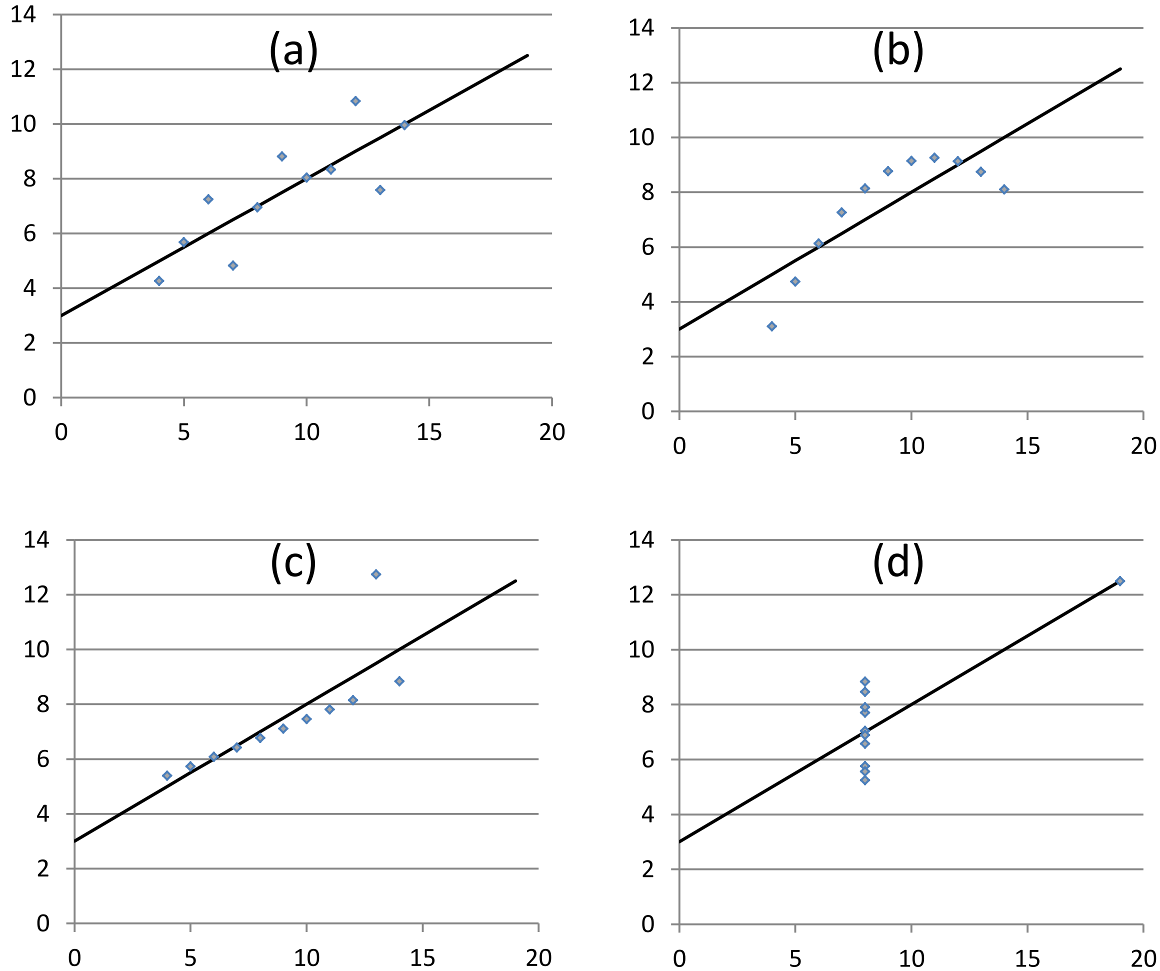


Figure A1:This figure illustrates the four datasets with identical statistical properties but different correlation patterns (Anscombe, 1973).

## Appendix B

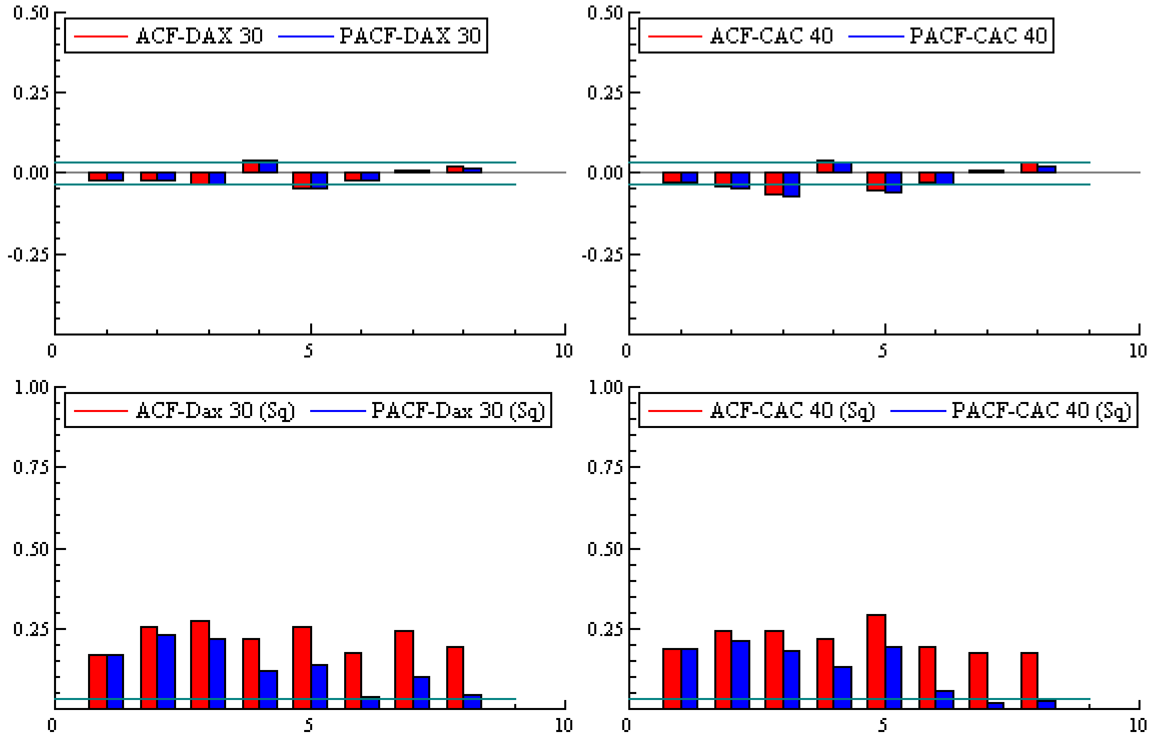


Figure BI: ACF and PACF plots of the log returns and the squared log returns of the DAX-30 and the CAC-40 stock indices. The ACF plot of the DAX 30 returns indicates the presence of some serial correlation at the 4th and the 5th lag. The same plot indicates that the DAX 30 returns may be modeled by an ARMA(5,5) model. The big spikes in the PACF plot of the squared returns of the DAX 30 index indicates that the DAX 30 returns are not serially independent and have some ARCH effect. The ACF plot of the CAC 40 returns indicates the presence of serial correlation in the first 5 lags. The ACF and PACF plot of the CAC 40 indicates that the return series may be well modeled by an ARMA(5,5). As in the case of the DAX 30 index, the ACF plot of the CAC 40 indicates the returns are not independent and exhibit some ARCH effect.

| Parameter | DAX-30 | | | | CAC-40 | | | |
|---|---|---|---|---|---|---|---|---|
| | Estimate | Error | T-statistic | P-value | Estimate | Error | T-statistic | P-value |
| $\mu$ | 0.001 | 0.000 | 3.802 | 0.000 | 0.001 | 0.000 | 3.172 | 0.002 |
| $\varphi_1$ | 0.427 | 0.050 | 8.576 | 0.000 | 0.613 | 0.113 | 5.423 | 0.000 |
| $\varphi_2$ | -0.905 | 0.033 | -27.590 | 0.000 | - | - | - | - |
| $\theta_1$ | -0.436 | 0.042 | -10.500 | 0.000 | -0.672 | 0.117 | -5.761 | 0.000 |
| $\theta_2$ | 0.920 | 0.035 | 26.410 | 0.000 | - | - | - | - |
| $\alpha_0$ | 0.000 | 0.006 | 3.153 | 0.002 | 0.000 | 0.005 | 3.175 | 0.002 |
| $\alpha_1$ | 0.090 | 0.011 | 8.385 | 0.000 | 0.087 | 0.011 | 7.885 | 0.000 |
| $\beta_1$ | 0.904 | 0.010 | 86.220 | 0.000 | 0.908 | 0.011 | 84.780 | 0.000 |
| $\nu$ | 10.401 | 2.171 | 4.790 | 0.000 | 10.744 | 2.048 | 5.246 | 0.000 |

Table BII: ARMA-GARCH-t estimation

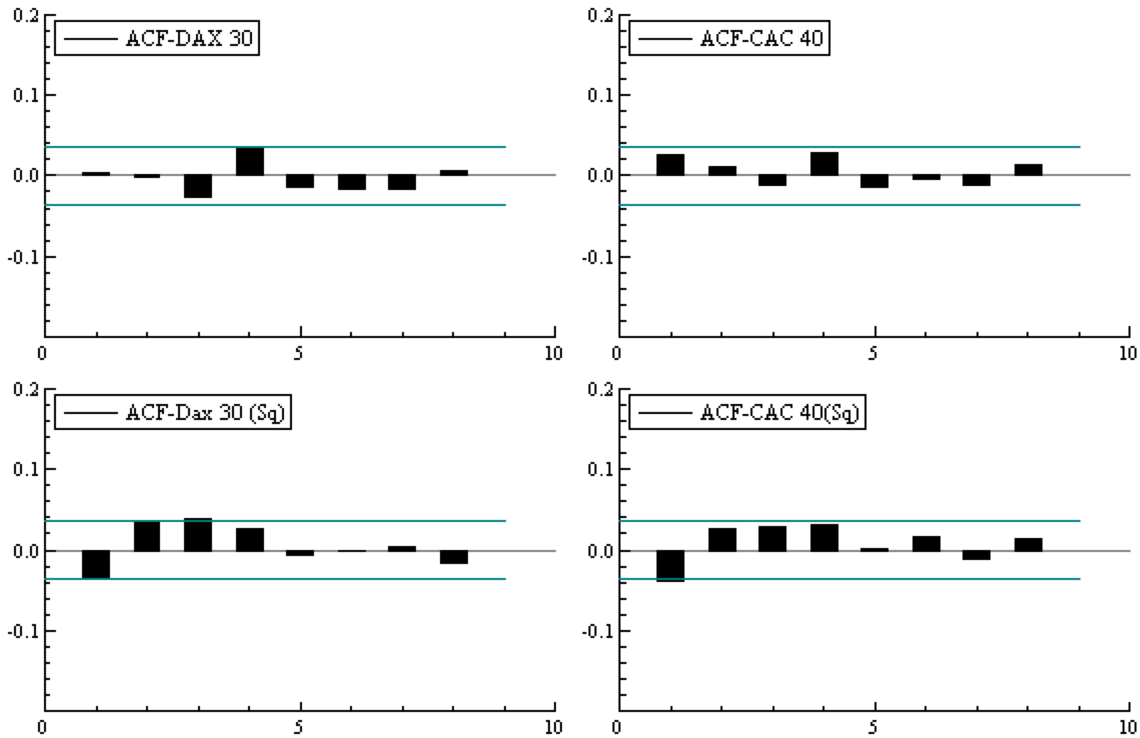


Figure BII : ACF plot of the standardized residuals and the squared standardized residuals (Sq) of the DAX-30 and the CAC-40 stock indices. The ACF plot of the DAX-30 and the CAC-40 standardized residuals indicate that the series are serially independent. The ACF plots of the squared standardized show no high spikes which indicate that the variance equations are well modeled by the GARCH-t model.

## Appendix C

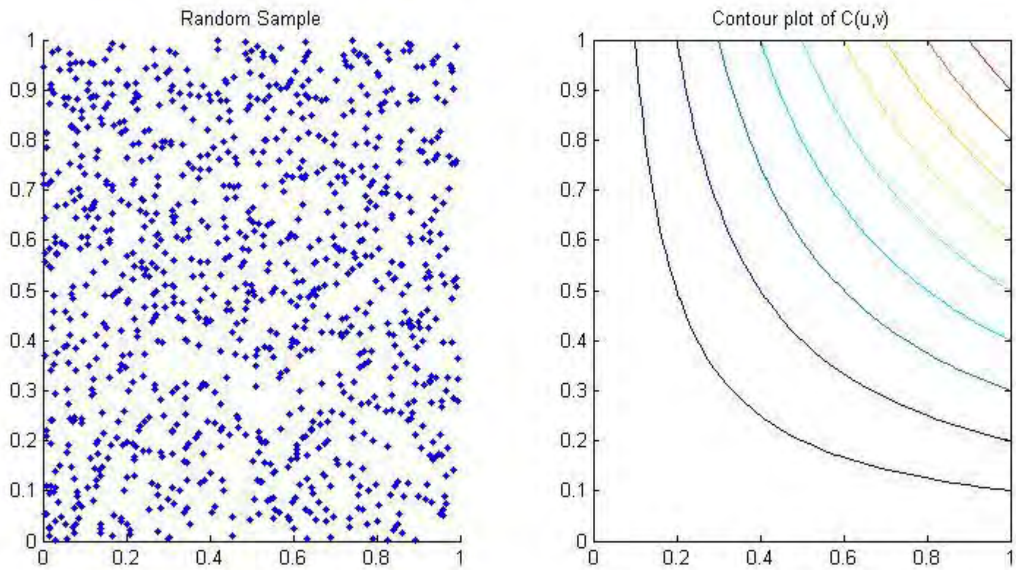


Figure CI: Scatter plot and contour plot of the product copula (case of independence)

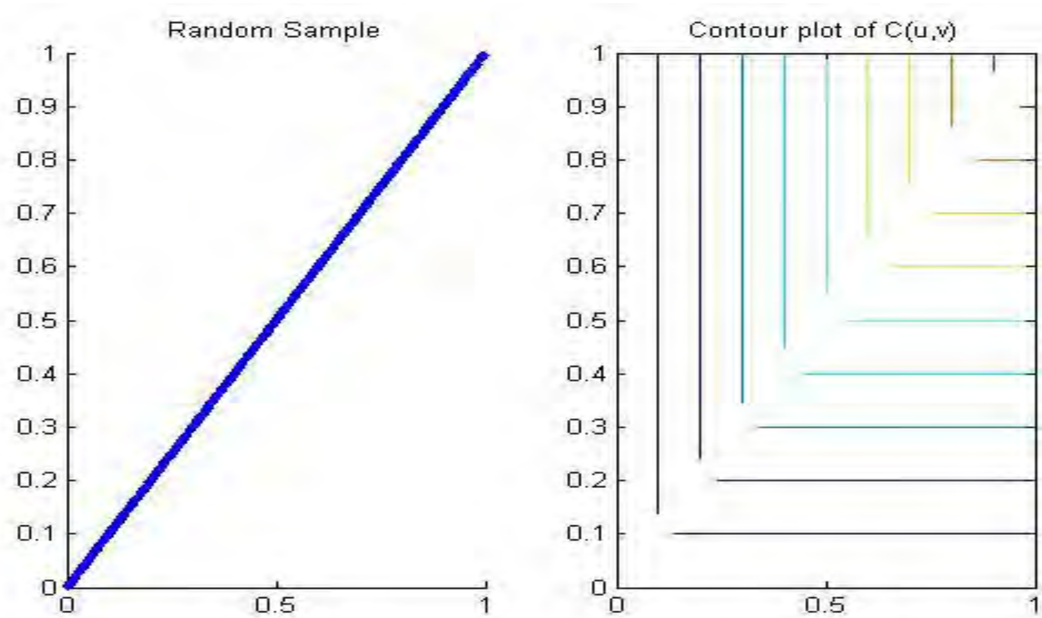


Figure CII: Scatter plot (1000 simulated values) and contour plot of the Frechet-Hoeffding upper bound estimated Gumbel (infinity)