Early Exercise Opportunities for American Call Options on Dividend-Paying Assets

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Abstract

American call options are contracts that give the right, but not the obligation, to buy an asset underlying the contract for a predetermined price at any time during the lifetime of the contract. A well known result of quantitative finance is that it's never optimal to exercise an American call option before the expiry date of the contract if the underlying asset does not pay dividend. When the underlying asset does pay dividend this changes. This paper examines in which situations its optimal to exercise the option on a dividend-paying asset before the contract expires. The two types of assets that will be discussed are assets with a continuous dividend yield and assets with a discrete dividend payment. Finding the particular value of these assets at which it is optimal to exercise the option early can be posed as a free boundary problem, where it is optimal to exercise the option early if the asset price falls above this boundary. In this paper early exercise opportunities for American options are found by analyzing the behavior of these free boundaries using both analytical and numerical methods.

Contents

1 Introduction

An European call/put option gives the right, but not the obligation, to buy/sell an asset at the expiry date T for strike price E. Fischer Black and Myron Scholes derived a formula to evaluate the price of an European option called *Black and Scholes Formula*[8]:

$$C_{eu}(S,t) = S\mathcal{N}(d_1) - Ee^{-r(T-t)}\mathcal{N}(d_2) \qquad \text{for a call option}$$
$$P_{eu}(S,t) = Ee^{-r(T-t)}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1) \qquad \text{for a put option}$$

These equations are derived from a *Partial Differential Equation* called the *Black and Scholes Differential Equation*. These closed from solutions can be found because the time until expiry is fixed, and early exercise is not allowed. An American option works in the exact same way, but the option can be exercised at any time between the settlement date of the contract until the expiry date $t_{\in}[0,T]$. Causing there to be an asset price S at each time $t_{\in}[0,T]$ for which it's optimal to exercise. This value of S is called the *optimal exercise boundary*. There is no prior knowledge on this *optimal exercise boundary* making pricing an American option a *free boundary problem*. By reformulating the American option pricing problem into a *linear complementarity problem* makes it possible to find the *free boundary* using numerical and asymptotic methods.

In the second *section* there will be a brief introduction in the economic theory behind option pricing. The third *section* shows why its not optimal to exercise the American option on a non dividend paying asset before expiry by using this theory. *Sections 4* and *5* define the properties of the two dividend paying assets and pose the *linear complementarity problem* for the American options on these assets to find the *free boundary*. In *section 6* the *linear complementarity problem* for both options is solved using *finite-difference methods*. The results of these methods are presented in *section 7*. In *section 8* an asymptotic solution to the *free boundary* is derived to analyze the behavior of the *free boundary* near the expiry date. The final *section* compares the results of the *finite-difference method* with the asymptotic solution. This paper will in large part rely and extend on the work done by Willmott, Howison and Dewynne in *the Mathematics of Financial Derivatives* [1].

2 Economic Theory Behind Option Pricing

There are two important economic theories underlying the process of pricing an option: the *efficient market hypothesis* and *arbitrage pricing theory*. This section covers the parts of both theories which are relevant to this paper.

2.1 Efficient Market Hypothesis

Option prices will obviously depend heavily on the fluctuations of the underlying asset. Fluctuations in asset prices are explained by the *efficient market hypothesis* (EMH). The EMH states that in a competitive market all market data regarding an asset is already reflected in the current price of the asset. For example, there would exist a forecasting method that would predict that an asset price is going to rise in value. Each investor would try to buy the asset before the price increases, causing a rapid increase in price. In other words a favorable forecast or news on *future* performance leads into an immediate change in the current asset price. If asset prices immediately change when new information arises and new information arrives randomly the asset prices should move randomly [2]. Wilmott et al. in [1] model the relative changes in the asset prices caused by the arrival of new information with a *random walk*:

$$\frac{dS}{S} = \sigma dX + \mu dt$$

Here S is the asset price, σ is the volatility of the asset and μ the average growth of the asset price. dX is a *Wiener process* which is a random variable with mean 0 and variance dt.

2.2 Arbitrage Pricing Theory

Option prices are often deduced by equating two economically equivalent portfolios. If two portfolios are economically equivalent, but they have different prices arbitrage opportunities arise. An Arbitrage opportunity is a situation in which an investor can make a risk-free profit larger than the risk-free rate. The risk-free rate is the return on a risk-free asset. Practitioners use Treasury bills or Government Bond yields [4] as proxy for this risk-free asset. *Arbitrage pricing theory* APT states that well-functioning markets do not allow long persistence of arbitrage opportunities [2].

Say, there exists an arbitrage opportunity caused by to two portfolios that are economically equivalent, but they have different prices. Each investor will take an as big as possible long position in the underpriced portfolio and short the same amount in the overpriced portfolio regardless of their risk aversity. This activity would drive the price of the underpriced portfolio up and the price of the overpriced portfolio down until they converge to the same price.

3 Early Exercise of American Call Options without Dividend

In the introduction it was mentioned that it's never optimal to exercise an American call option if the underlying asset does not pay dividend. Berk and DeMarzo in [3] give an intuitive explanation of this phenomenon using APT and the *Put-Call Parity*.

3.1 Put-Call parity

European put options give the right to sell an asset at time T for a price E, thus at expiry the option pays out $MAX\{E - S, 0\}$. Holding a portfolio of underlying asset and a European call option will protect the investor from situations where the asset price falls below E. This portfolio is called the *protective put* and it's payoff is $S + MAX\{E - S, 0\} = MAX\{E, S\}$. European Call options give the right to buy an asset at time T for a price E, thus at expiry the option pays out $MAX\{S - E, 0\}$. Holding a portfolio of the European call option and a *zero coupon bond* that matures at time T with a *face value* of E will ensure that the investor raises enough money to buy the asset at expiry. This portfolio is called the *covered call* and it's payoff at time T is $E + MAX\{S - E, 0\} = MAX\{S, E\}$.

Since these portfolios are equivalent they should have the same price. If this is not the case there will be an arbitrage opportunity. Equating the prices of these portfolios gives the *Put-Call Parity*:

$$C_{eu}(S, t) + Ee^{-r(T-t)} = P_{eu}(S, t) + S$$

In figure 1 the payoffs of these portfolios are graphically shown.

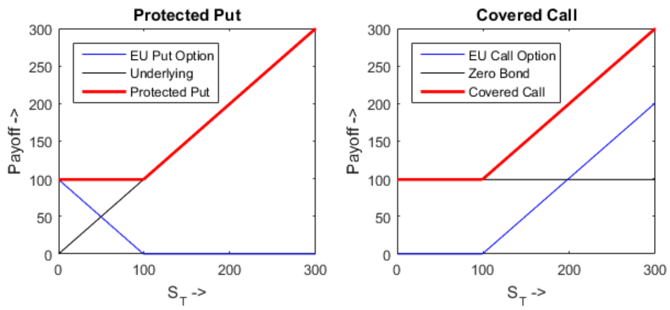


Figure 1: On the right the Covered Call and on the left the Protected Put both on the same asset and both options have the same exercise price $E = 99$ and $T = 1$ (one year).

3.2 Adding an Early Exercise Opportunity to the European Call Option

As discussed in *chapter 1* American options can be exercised at any time $t_\in [0, T]$. Having an early exercise opportunity on an option is only beneficial if the option value is smaller or equal to the payoff. If the option price is greater than the payoff an investor would always sell the option rather then exercising it. To look for this situation for an European call option observe the *Put-Call parity*:

$$C_{eu}(S,t) + Ee^{-r(T-t)} = P_{eu}(S,t) + S$$
$$\text{Define } int_{r,t} = E - Ee^{-r(T-t)}$$
$$C_{eu}(S,t) = \underbrace{S - E}_{\text{payoff}} + \underbrace{int_{r,t} + P_{eu}(S,t)}_{\text{time value}}$$

The value of the European call option is composed of the time value and the payoff. An investor will only exercise an option if the payoff is greater then 0, $S - E > 0$. The time value consists of the value of the put option on the same asset with the same strike price and expiry date and the $int_{r,t}$ term. The price of the put option is always greater or equal to 0 and if interest rates are positive $int_{r,t} = E - Ee^{-r(T-t)}$ is also greater then 0. This makes the value of a call option always greater then the payoff. Thus adding an early exercise opportunity to an European call option does not add any value to the option if the underlying asset does not pay any dividend. Because of this the prices of the American option and the European option are equal, $C_{eu}(S,t) = C_{am}(S,t)$. If this would not be the case there would be an arbitrage opportunity. Investors would buy the cheaper one and write the expensive one and gain a risk-free profit.

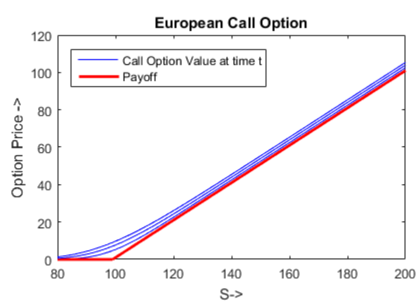


Figure 2: European call option with strike price $E = 99$, $r = 0.06$, $T = 1$ and $\sigma = 0.2$ at different times t before expiry.

4 Dividend Paying Assets

Before looking into options the dynamics of dividend paying assets need to be clearly defined. There are several different dividend payment structures. As mentioned in the introduction this paper will go into detail on two different payment structures: one discrete dividend payment and a continuous dividend yield. A dividend payment reduces the price of an asset. If this would not be the case an investor would buy the asset, receive the dividend payments and sell it right after giving the investor a risk-free profit. To incorporate the dynamics of price changes caused by dividend payments the asset price model described in *section 2.1* is modified by subtracting a $D(S,t)$ term from the average growth μ, where $D(S,t)$ is a function that represents on the dividend payment structure of the asset:

$$dS = \sigma S dX + (S\mu - D(S,t))dt$$

4.1 A Single Discrete Dividend Payment

Many companies make dividend payments to their shareholders. The dates of these payments are known and should be treaded discretely [1]. This paper will go into detail if there is one single dividend payment

Consider an underlying asset with the following dividend payment structure: During the lifetime of the option the asset pays a percentage, y_d, of the asset price at time $t = t_d$ as dividend. The date and this percentage are known. Assuming efficient markets there should be a discontinuous drop in the asset price of y_d percent at t_d. Mathematically this gives a jump $S_{t_d^+} = S_{t_d^-}(1-d_y)$, where t_d^- and t_d^+ are the moments right before and after the dividend date t_d respectively. A discontinuous jump at time t_d can be added to a differential equation by using the shifted Dirac delta function, $\delta(t-t_d)$. Using this, the function $D(S,t)$ should be of the form $D_\delta S\delta(t-t_d)$, where D_δ is a constant that is related to the percentage change y_d. The D_δ is fitted to y_d by integrating across the dividend date:

$$\begin{aligned} dS &= \sigma S dX + (S\mu - D_\delta S\delta(t-t_d))dt \\ \int_{t_d^-}^{t_d^+} \frac{dS}{S} &= \int_{t_d^-}^{t_d^+} \sigma dX + \int_{t_d^-}^{t_d^+} \mu dt - D_\delta \int_{t_d^-}^{t_d^+} \delta(t-t_d)dt \end{aligned}$$

The moments in time t_d^- and t_d^+ only differ infinitesimally causing the integral involving μ to be zero. The integral regarding the *Wiener process* will be equal to zero as well, because it gives a normally distributed random variable with mean 0 and variance $\sigma^2 dt$. In this integral we pass an infinitesimal point in time this will be a normally distributed variable with mean 0 and variance 0. Leaving only the integral with the delta function.[1]

$$\begin{aligned} \int_{t_d^-}^{t_d^+} d\ln(S) &= -D_\delta \int_{t_d^-}^{t_d^+} d\mathcal{H}(t-t_d) \\ \ln\left(S_{t_d^+}\right) - \ln\left(S_{t_d^-}\right) &= -D_\delta(\mathcal{H}(t_d^+ - t_d) - \mathcal{H}(t_d^- - t_d)) \\ S_{t_d^+} &= S_{t_d^-} e^{-D_\delta} \\ D_\delta &= -\ln(1-d_y) \end{aligned}$$

Thus the dynamics of the asset are given by $dS = S\sigma dX + S(\mu + \delta(t-t_d)\ln(1-d_y))dt$.

[1] A complete mathematical explanation of why this is the case requires measure theory which falls out of the scope of this paper. For a more detailed explanation see Measures, Integrals and Martingales by René L. Schilling [5].

4.2 A Continuous Dividend Yield

The holder of an asset with a continuous dividend yield receives $D_0 S dt$ dividend over dt time, where D_0 is the continuous dividend yield. Plugging this into the differential equation gives $dS = S\sigma dX + S(\mu - D_0)dt$. Hull in [6] and Paul Wilmott et al. in [1] both give two applications of options on this asset model:

- **A Foreign Exchange (FX) option**
  The asset underlying the FX option, S, is the exchange rate between domestic currency and foreign currency. A FX call option give the possibility to buy foreign currency using exchange rate E. In this model r will denote the domestic risk-free rate and the dividend yield D_0 denotes the foreign risk-free rate.

- **An option on an index**
  An index, like the AEX and the S&P 500, is composed of many different shares paying dividend at different times. Rather than modeling this as a succession of discrete payments one could view them as a continuous stream of payments.

The *Black and Scholes Differential Equation* and the formula for the price for a European option on an asset with a constant dividend yield is derived in *appendix II*. In figure 3 a European call FX option with a strike price $E = 1.1$, domestic risk-free rate $r = 0.06$ and foreign risk-free rate $D_0 = 0.05$ is shown. The strike price is in this case the exchange rate at which the holder of the option can buy foreign currency. Contrary to the call option without dividend the European call in figure 3 can fall below the the payoff.

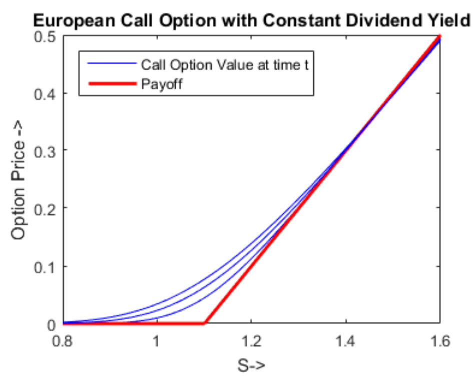


Figure 3: European call option with strike price $E = 1.1$, $r = 0.06$, $T = 1$, $\sigma = 0.2$ and $D_0 = 0.05$ at different times t before expiry

5 American options on Dividend Paying Assets

In this section the American options on the dividend paying assets defined in *section 4* are discussed. An American option can be exercised at any time. This makes it impossible for the option value to be worth less then its payoff, because else it gives way to arbitrage opportunities. If some American option ,$V_{am}(S, t)$, would be less then its payoff an investor would buy the option and immediately exercise it giving him/her a risk-free profit. This gives American call options the following constraint $C_{am}(S, t) \geqslant S - E$. In figure 3 it is shown that for an European option on a dividend paying asset there exists an asset price S such that the option price falls below the payoff. This causes the American call to hit the constraint and become $S - E$, making exercising the option preferable over holding it. The minimal value of S for which this is the case is called the *optimal exercise price* and will be denoted as $S_f(t)$. There is no prior knowledge on the behavior of $S_f(t)$ for this reason it's called a *free boundary* [1]. This section will give a mathematical framework called a *linear complementarity problem* in which it is possible to analyze the behavior of this *free boundary*.

5.1 American Options on a Asset with a Continuous Dividend Yield

The previously mentioned constraint causes a violation in *Black and Scholes Differential Equation* if it's optimal to exercise an American Option. If $S \geqslant S_f(t)$ at time t if it's optimal to exercise and thus the value of the option becomes: $C_{am}(S,t) = S - E$. This causes the equality in the *Black and Scholes Differential Equation* to come an inequality $\frac{\partial C_{eu}}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C_{eu}}{\partial S^2} + (r - D_0)S\frac{\partial C_{eu}}{\partial S} - rC_{eu}(S,t) \leqslant 0$. $S_f(t)$ is the contact point of the *Black and Scholes Differential Equation* and $MAX\{S - E, 0\}$. This leads to the following expression:

$$S < S_f(t)$$
$$\frac{\partial C_{am}}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C_{am}}{\partial S^2} + (r - D_0)S\frac{\partial C_{am}}{\partial S} - rC_{am}(S,t) = 0$$
$$C_{am}(S,t) \geqslant MAX\{S - E, 0\}$$

$$S \geqslant S_f(t)$$
$$\frac{\partial C_{am}}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C_{am}}{\partial S^2} + (r - D_0)S\frac{\partial C_{am}}{\partial S} - rC_{am}(S,t) \leqslant 0$$
$$C_{am}(S,t) = MAX\{S - E, 0\}$$

$$C_{am}(S_f(t),t) = S_f(t) - E \qquad \frac{\partial C_{am}(S_f(t),t)}{\partial S} = 1$$
$$\text{with } C_{am}(S,t) \text{ and } \frac{\partial C_{am}}{\partial S} \text{ continuous}$$

This expression can be reformulated into a *linear complementarity problem*:

$$\left(\frac{\partial C_{am}}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C_{am}}{\partial S^2} + (r - D_0)S\frac{\partial C_{am}}{\partial S} - rC_{am}(S,t)\right)(C_{am}(S,t) - S + E) = 0$$
$$\frac{\partial C_{am}}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C_{am}}{\partial S^2} + (r - D_0)S\frac{\partial C_{am}}{\partial S} - rC_{am}(S,t) \leqslant 0$$
$$C_{am}(S,t) \geqslant S - E$$

The *Black and Scholes Differential Equation* is solved by calculating the discounted expected payoff of the option against the risk-free rate. Wilmott et al. model this as a diffusion process by transforming the *Black and Scholes Differential Equation* into the *Heat Equation* using the following variable transformation:

$$C_{am}(S,t) = Eu(x,\tau)e^{\alpha x + \beta \tau}$$
$$t = T - \frac{\tau}{\sigma^2/2}$$
$$S = Ee^x$$
$$\alpha = -\frac{1}{2}(k' - 1) \text{ and } \beta = -\frac{1}{4}(k' - 1)^2 - k$$
$$k' = \frac{r - D_0}{\sigma^2/2} \text{ and } k = \frac{r}{\sigma^2/2}$$

Applying this transformation to $\left(\frac{\partial C_{am}}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C_{am}}{\partial S^2} + (r - D_0)S\frac{\partial C_{am}}{\partial S} - rC_{am}(S,t)\right)$ reduces it to $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$. In this transformation t was set to $T - \tau/\frac{\sigma^2}{2}$ implying that when $\tau = 0$, t is equal to T. At time T the option is at expiry, thus the option is at the payoff. In the derivation of the European Call option in *Appendix II* this transformation is shown in greater detail. The free boundary in this transformation changes to: $x_f(\tau) = \ln\left(\frac{S_f(t)}{E}\right)$. The early exercise payoff in terms of the financial variables is $C_{am}(S,t) = MAX\{S - E, 0\}$. The early exercise payoff in terms of the *Heat Equation* will be denoted as $g(x,\tau)$:

$$\begin{aligned} C_{am}(S,t) &= MAX\{S - E, 0\} \\ Eg\,(x,\tau)\,e^{-x\frac{k'-1}{2} - \tau\left(\frac{(k'-1)^2}{4} + k\right)} &= MAX\{Ee^x - E, 0\} \\ g(x,\tau) &= e^{\tau\left(\frac{(k'-1)^2}{4} + k\right)} MAX\{e^{x\frac{k'+1}{2}} - e^{x\frac{k'-1}{2}}, 0\} \end{aligned}$$

Now all the ingredients are gathered it is possible to transform the *linear complementarity problem* in terms of the *Heat Equation*:

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) = 0$$
$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \geqslant 0$$
$$(u(x,\tau) - g(x,\tau)) \geqslant 0$$
$$g(x,0) = u(x,0)$$
$$\text{With } u(x,\tau) \text{ and } \frac{\partial u}{\partial x}(x,\tau) \text{ continuous}$$

In *section 6* this problem will be solved numerically using a *finite difference method*.

5.2 American Options with One Discrete Dividend Payment

In *section 4.1* it was shown that the dividend payment of y_d percent of the asset price at dividend date t_d caused a jump in the stock price $S_{t_d^+} = S_{t_d^-}(1 - d_y)$. To avoid arbitrage opportunities the option price should be continuous in time when moving across the dividend date. To make this possible the following should hold:

$$\begin{aligned} C_{am}(S_{t_d^+}, t_d^+) &= C_{am}(S_{t_d^-}, t_d^-) \\ C_{am}(S(1 - d_y), t_d^+) &= C_{am}(S, t_d^-) \end{aligned}$$

This is called a *jump condition* and makes the option price change discontinuously for a fixed S when crossing the dividend date, but it makes the option price a continuous function in time for each realization of the asset's random walk[1].

In *section 3* it was shown that $C_{eu}(S,t) = C_{am}(S,t)$ if the underlying asset does not pay any dividend. This is of great relevance when evaluating an American option with discrete dividend payments, because if the option is bought after the dividend date it will, obviously, behave in the same way. When bought before the dividend date the payoff at expiry is $MAX\{S(1 - d_y) - E, 0\}$. By using the fact that $(1 - y_d)$ is a scaling of S, which leaves *Black and Scholes Equation* invariant, it is possible to change the payoff into: $(1 - d_y)MAX\{S - \frac{E}{1-d_y}\}$ [1]. This translates the drop in asset price to a jump in the exercise price, thus what's left is the exact same *linear complementary problem* as an American option without dividend as described by Wilmott et al., but with an discontinuous early exercise payoff:

$$
\begin{aligned}
C_{am}(S,t) &= (1-d_y)MAX\{S - \frac{E}{1-d_y}\} \text{ if } S \geqslant S_f(t) \text{ and } t \geqslant t_d \\
C_{am}(S,t) &= MAX\{S-E,0\} \qquad\qquad \text{if } S \geqslant S_f(t) \text{ and } t < t_d
\end{aligned}
$$

Using the transform to the *Heat Equation* gives:

$$
\begin{aligned}
&g(x,\tau) = e^{\tau\left(\frac{(k+1)^2}{4}\right)} MAX\{e^{x\frac{k+1}{2}} - e^{x\frac{k-1}{2}}, 0\} \qquad\qquad \text{if } \tau > \tau_d \\
&g(x,\tau) = (1-y_d)e^{\tau\left(\frac{(k+1)^2}{4}\right)} MAX\{e^{x\frac{k+1}{2}} - e^{x\frac{k-1}{2}}/(1-y_d), 0\} \text{ if } \tau \leqslant \tau_d \\
&g(x,0) = u(x,0)
\end{aligned}
$$

The τ_d represents the transformed dividend date: $(T-t_d)\frac{\sigma^2}{2}$.

6 Finite-Difference Method for the Heat Equation

The *finite-difference method* is a class of numerical methods that is used to solve *differential equations* and *linear complementary problems* using difference equations. In this section the *finite-difference method* will be used to solve the *linear complementary problems* stated in *section 5*. After solving the problems in terms of the *Heat Equation* the results will be transformed back into financial variables. The *finite-difference method* uses discretized derivatives with respect to x and τ to solve the *Heat Equation* for $u(x,\tau)$.

$$
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}
$$

There are multiple ways to take discretized derivatives. A forward, a backward and a central difference approximation. For the derivative with respect to x a central difference method is used by taking a forward followed by a backwards difference difference approximation:

$$
\begin{aligned}
\frac{\partial u}{\partial x} &\approx \frac{u(x+\delta x,\tau) - u(x,\tau)}{\delta x} \\
\frac{\partial^2 u}{\partial x^2} &\approx \frac{\frac{u(x+\delta x,\tau)-u(x,\tau)}{\delta x} - \frac{u(x,\tau)-u(x-\delta x,\tau)}{\delta x}}{\delta x} = \frac{u(x+\delta x,\tau) - 2u(x,\tau) + u(x-\delta x,\tau)}{(\delta x)^2}
\end{aligned}
$$

For the derivative with respect to τ only the forward and backward approximation are used:

$$
\begin{aligned}
\text{Forward difference in time: } \frac{\partial \tau}{\partial x} &\approx \frac{u(x,\tau+\delta\tau) - u(x,\tau)}{\delta \tau} \\
\text{Backward difference in time: } \frac{\partial \tau}{\partial x} &\approx \frac{u(x,\tau) - u(x,\tau-\delta\tau)}{\delta \tau}
\end{aligned}
$$

Using the forward difference with respect to τ and the central difference with respect to x is called the *explicit method*. The backward difference with respect to τ and the central difference with respect to x is called the *implicit method*.

$$\text{Explicit Method: } \frac{u(x,\tau+\delta\tau)-u(x,\tau)}{\delta\tau} = \frac{u(x+\delta x,\tau)-2u(x,\tau)+u(x-\delta x,\tau)}{(\delta x)^2}$$
$$\text{Implicit Method: } \frac{u(x,\tau)-u(x,\tau-\delta\tau)}{\delta\tau} = \frac{u(x+\delta x,\tau)-2u(x,\tau)+u(x-\delta x,\tau)}{(\delta x)^2}$$

The eventual *finite-difference method* used is called the *Crank-Nickolson method*, which is the average of a *explicit method* and a *implicit method*. Before defining the *Crank-Nickolson method* we look into this discretization.

6.1 Setting up the Finite-Difference Method

The x-axes and τ-axes of the (x,τ) plain are divided into a finite amount of steps of size δx and $\delta\tau$. The space variable x in the *Heat Equation* is transformed by $S = Ee^x$ since $S_\in[0,\infty)$, $x_\in(-\infty,\infty)$, since a computer cannot separate an infinite amount of steps two big integers are used such that: $N^-\delta x \leqslant x \leqslant N^+\delta x$. Let M be the amount of steps in the τ direction since the transformed lifetime of the option, $\frac{T\sigma^2}{2}$, is finite this won't cause any problems. This discretization divides the (x,τ) plain into a mesh, see figure 4. In this mesh $u(x,\tau)$ is only calculated at the mesh points $(m\delta\tau, n\delta x)$, define: $u_n^m = u(n\delta x, m\delta\tau)$.

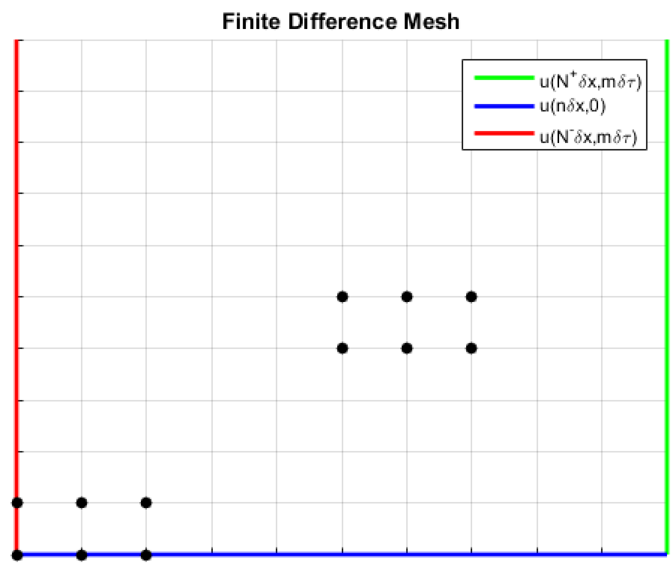


Figure 4: The finite-difference mesh

u_n^m is the value of $u(x,\tau)$ evaluated at $(m\delta\tau, n\delta x)$. Using the limits of n and m its is possible to define the boundary and initial conditions. The boundary conditions are obtained by looking at the boundaries of the call option: $C_{am}(S,t)$ behaves like $S-E$ when S is going to ∞ and $C_{am}(0,t) = 0$. This makes the boundary condition in terms of the *Heat Equation* $u_{N^-}^m = g(N^-\delta x,\tau)$ and $u_{N^+}^m = g(N^+\delta x,\tau)$.[2] These values

[2] $g(x,\tau)$ is the early exercise boundary from *section 5*

can be calculated before the algorithm and are known throughout the entire algorithm. The initial condition, $u(x,0)$, is also known before starting the algorithm. This is the transformed payoff of the option. These conditions are highlighted in figure 4.

6.2 The Crank-Nickolson Method

By defining $\alpha = \frac{\delta\tau}{(\delta x)^2}$ the explicit and implicit method are rewritten to:

$$\begin{aligned}
\text{Explicit Method: } u_n^{m+1} &= \alpha u_{n+1}^m + (1-2\alpha)u_n^m + \alpha u_{n-1}^m \\
\text{Implicit Method: } u_n^m &= -\alpha u_{n+1}^{m+1} + (1+2\alpha)u_n^{m+1} - \alpha u_{n-1}^{m+1}
\end{aligned}$$

The *Crank-Nickolson method* takes the average of these two:

$$u_n^{m+1} - \frac{\alpha}{2}\left(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n+1}^{m+1}\right) = u_n^m + \frac{\alpha}{2}\left(u_{n+1}^m - 2u_n^m + u_{n+1}^m\right) \qquad (\star)$$

The Crank-Nickolson method takes 3 values at timestep m to calculate the the 3 values at $m+1$, see the black dots in figure 4. This makes the entire right hand side of $\star$ known at $m+1$ take $Z_n^m = u_n^m + \frac{\alpha}{2}\left(u_{n+1}^m - 2u_n^m + u_{n+1}^m\right)$. The method starts from $m=0$, where the values are known from the initial condition. At timestep $m+1$ the values $u_{N^+}^{m+1}$ and $u_{N^-}^{m+1}$ are known aswell from the boundary conditions.

$$\begin{aligned}
u_n^{m+1} - \frac{\alpha}{2}\left(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n+1}^{m+1}\right) &= Z_n^m \\
(1-\alpha)u_n^{m+1} - \frac{1}{2}\alpha\left(u_{n-1}^{m+1} + u_{n+1}^{m+1}\right) &= Z_n^m
\end{aligned}$$

When calculating $u_{N^-+1}^{m+1}$ the algorithm wil use the known value from the boundary condition $u_{N^-}^{m+1}$. The same goes for $u_{N^+}^{m+1}$ when calculating $u_{N^+-1}^{m+1}$. This leads to the following expression:

$$\begin{cases}
(1-\alpha)u_n^{m+1} - \frac{1}{2}\alpha\left(u_{n+1}^{m+1}\right) & = Z_n^m + \frac{\alpha}{2}u_{N^-}^{m+1} \text{ if } n = N^- + 1 \\
(1-\alpha)u_n^{m+1} - \frac{1}{2}\alpha\left(u_{n-1}^{m+1} + u_{n+1}^{m+1}\right) & = Z_n^m \qquad \text{ if } N^- + 1 < n < N^+ - 1 \\
(1-\alpha)u_n^{m+1} - \frac{1}{2}\alpha\left(u_{n-1}^{m+1}\right) & = Z_n^m + \frac{\alpha}{2}u_{N^+}^{m+1} \text{ if } n = N^+ - 1
\end{cases}$$

This expression gives linear system of $N^+ - N^- - 1$ equations and $N^+ - N^- - 1$ unknowns.

$$\underbrace{\begin{bmatrix} 1+\alpha & -\frac{\alpha}{2} & 0 & \dots & 0 \\ -\frac{\alpha}{2} & 1+\alpha & -\frac{\alpha}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1+\alpha \end{bmatrix}}_{A} X \underbrace{\begin{bmatrix} u_{N^-+1}^{m+1} \\ u_{N^-+2}^{m+1} \\ \vdots \\ u_{N^+-1}^{m+1} \end{bmatrix}}_{u^{m+1}} = \underbrace{\begin{bmatrix} Z_{N^-+1}^m \\ Z_{N^-+2}^m \\ \vdots \\ Z_{N^+-1}^m \end{bmatrix} + \frac{\alpha}{2}\begin{bmatrix} u_{N^-}^{m+1} \\ 0 \\ \vdots \\ u_{N^+}^{m+1} \end{bmatrix}}_{b^m}$$

This leaves the matrix equation $Au^{m+1} = b^m$ which can be solved for u^{m+1} using a number of different methods.

6.3 Successive Over-Relaxation

The method used to solve the linear system is $Au^{m+1} = b^m$ is *Successive Over-Relaxation* (SOR). SOR loops through the vector u^{m+1} iteratively updating each u_n^{m+1} until convergence. Define $u_n^{m+1,k}$ as the k-th iteration of u_n^{m+1}. To start the iteration an initial guess is required for the values in u^{m+1}. Since the algorithm takes small steps in time the values in the known vector u^m are really close to u^{m+1}, thus for the initial guess $u_n^{m+1,0}$ the value u_n^m is used. During these iterations the early exercise boundary, $g(x,\tau)$, comes into play. At each iteration it is checked whether it's optimal to exercise by taking the maximum of $u_n^{m+1,k}$ and $g(n\delta x,(m+1)\delta\tau) = g_n^{m+1}$. To properly define the iterative process each individual equation in $Au^{m+1} = b^m$ is rewritten to:

$$u_n^{m+1} = \frac{1}{1+\alpha}\left(b_n^m + \frac{\alpha}{2}(u_{n-1}^{m+1} + u_{n+1}^{m+1})\right)$$

SOR starts at $u_{N^-+1}^{m+1,0}$, thus when it arrives at $u_{N^-+2}^{m+1,0}$ the value of $u_{N^-+1}^{m+1,1}$ is already calculated and can be used in the calculation of $u_{N^-+2}^{m+1,1}$. More general at $(k+1)$-th itteration of u_n^{m+1}, $u_n^{m+1,k+1}$, the value of $u_{n-1}^{m+1,k+1}$ is used instead of $u_{n-1}^{m+1,k}$. Further more SOR speeds up the convergence by multiplying the correction done on u_n^{m+1} at each itteration by *over-relexation parameter*, ω.

$$u_n^{m+1,k+1} = MAX\left(\left(u_n^{m+1,k} + \omega\left(\frac{1}{1+\alpha}\left(b_n^m + \frac{\alpha}{2}(u_{n-1}^{m+1,k+1} + u_{n+1}^{m+1,k})\right) - u_n^{m+1,k}\right)\right), g_n^{m+1}\right)$$

The SOR method used to solve the *linear complementarity problems* in *section 5* ω is set to, the somewhat trivial value, 1.[3] Leaving the following inteative method:

$$u_n^{m+1,k+1} = MAX\left(\frac{1}{1+\alpha}\left(b_n^m + \frac{\alpha}{2}(u_{n-1}^{m+1,k+1} + u_{n+1}^{m+1,k})\right), g_n^{m+1}\right)$$

This SOR method runs through the entire grid from $m = 0$ to $m = M$. After calculating the option value at each grid point the *Free Boundary* is found by evaluating the following expression at each timestep m:

$$x_f(m\delta\tau) = MIN(n\delta x \mid u_n^m = g_n^m \cup Ee^{n\delta x} > 0)$$

This returns the minimal asset price for which it is optimal to exercise the option. The condition $Ee^{n\delta x} > 0$ is added because if $S = Ee^x = 0$, $C_{am}(S,t) = 0$. This makes $u_n^m = g_n^m$, but at this value there is no early exercise opportunity.

7 Results Finite-Difference Method

In this section the *finite-difference method* is used on the two different American call options discussed in this paper.

7.1 Constant Dividend Yield

In this section we look at the American variant of the FX option shown in figure 3 with $E = 1.1$, $r = 0.06$, $D_0 = 0.05$, $T = 1$ and $\sigma = 0.2$. In figure 5 the results of the *finite-difference method* are shown.

[3]Setting ω to 1 defeats the purpuse of introducing an *over-relexation parameter*, because at $\omega = 1$ the two $u_n^{m+1,k}$ terms cancel. The SOR method with $\omega = 1$ is called the *Gauss-Seidel method*[1]

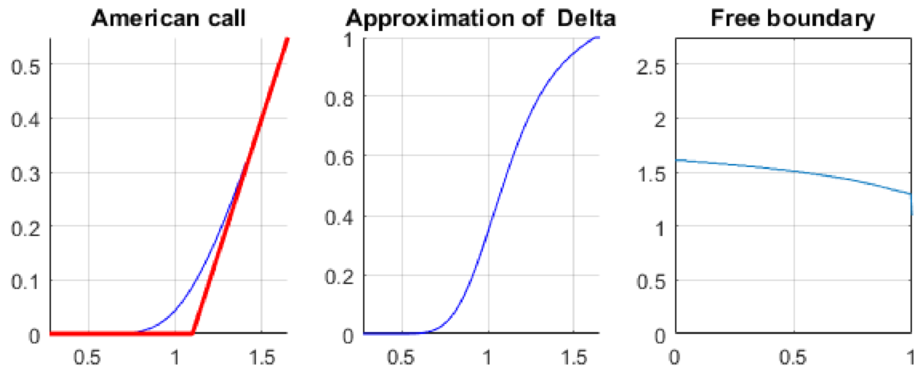


Figure 5: Results *finite-difference method* for an American Option with an constant dividend yield. The plot on the left shows the option value (blue) and the payoff (red). The center plot gives the $\Delta = \frac{\partial C}{\partial S}$ and the plot on the right shows the free boundary.

On the left the value of the call option at time $t = 0$ is shown. This plot shows the option value never falls below the constraint $C_{am}(S,t) \geqslant S - E$. On the right the *free boundary* is shown. The *free boundary* starts at expiry. At expiry it is obviously optimal to exercise if $S > E$. This is clearly shown in figure 5, where at $t = T$ the *free boundary* is at $S = 1.1$. While the boundary moves away from expiry it jumps up to a certain value and gradually increases from there. In figure 6 the *free boundaries* on the same option with different dividend yields are shown. For each option we see the exact same behavior when moving away from expiry. This behavior will be discussed in *section 8*. When the dividend yield is set to 0 the option becomes equivalent to the American option without dividend payments. In the plot on the top left of figure 6 the *free boundary* jumps up right after expiry and then never comes down. This is consistent with the result obtained *section 3* that its never optimal to early exercise an American option if the underlying does not pay dividend.

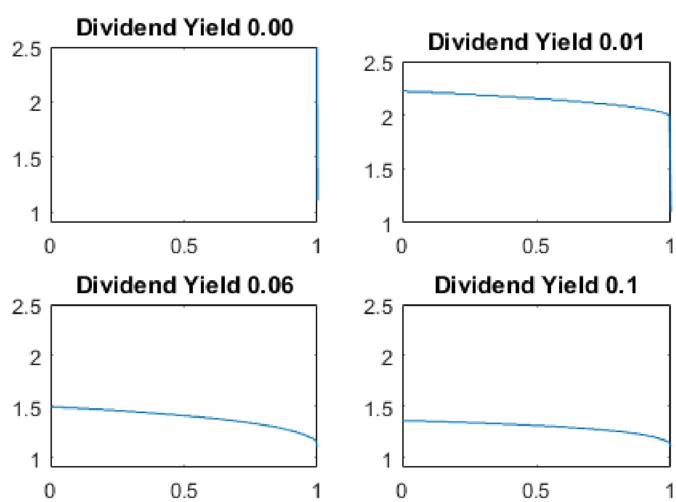


Figure 6: Free boundaries for the same option as in figure 5, but with different yields.

7.2 Discrete Dividend Payment

Hull in [6] states that for an American option on an asset with one dividend payment in its life time its only optimal to exercise right before the dividend payment. In figure 7 the results of the *finite-difference method* of an American option with one dividend payment are shown. This is the American variant of the option shown in figure 2 in *section 3* with an added dividend payment $y_d = 0.1$ at dividend date $t_d = 0.5$. At expiry the *free boundary* is equal to the exercise price and when it moves away from expiry it jumps up just as in the continuous case without dividend. This is an obvious result, because the option does not pay any dividend after the dividend payment at $t_d = 0.5$. Then figure 7 shows, just as Hull said, that the *free boundary* drops down right before the dividend date $t_d = 0.5$ and jumps up right after. This is caused by the jump condition introduced in *section 4.2*. In *section 4.2* the drop in asset price was translated to a jump in exercise price. Thus the exercise price at expiry is higher then the one at $t_\in[0, t_d)$. This causes the asset price to hit the payoff at t_d^-, just before the dividend date t_d. A second consequence of the jump is that the asset price does not hit the payoff at any moment before t_d^-.

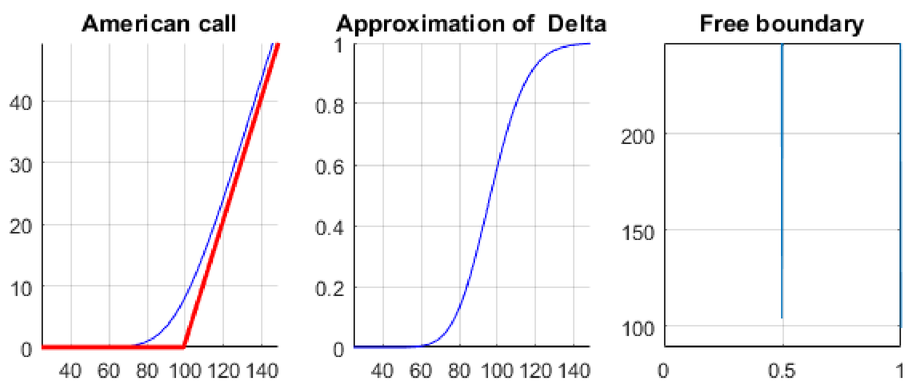


Figure 7: Results *finite-difference method* for an American Option with one discrete dividend payment.

8 An Asymptotic Solution to the Free Boundary

In *section 7* it was observed that right before expiry the *free boundary* of the call option with continuous dividend payments expressed a downwards jump to the exercise price E at expiry. Willmott et al. give an asymptotic solution for the free boundary to explain this behavior when $0 < D_0 < r$. This *section* gives a walk-through on how Willmott et al. reach this asymptotic solution. This asymptotic approach starts at the same problem statement as at the start of *section 6.1*, before it was reformulated to a *linear complementary* *problem*:

$$
\begin{aligned}
&S < S_f(t)\\
&\frac{\partial C_{am}}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C_{am}}{\partial S^2} + (r - D_0)S\frac{\partial C_{am}}{\partial S} - rC_{am}(S,t) = 0\\
&C_{am}(S,t) \geqslant MAX\{S-E, 0\}
\end{aligned}
$$

$$
\begin{aligned}
&S \geqslant S_f(t)\\
&\frac{\partial C_{am}}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C_{am}}{\partial S^2} + (r - D_0)S\frac{\partial C_{am}}{\partial S} - rC_{am}(S,t) \leqslant 0\\
&C_{am}(S,t) = MAX\{S-E, 0\}
\end{aligned}
$$

$$
C_{am}(S_f(t),t) = S_f(t) - E \qquad\qquad \frac{\partial C_{am}(S_f(t),t)}{\partial S} = 1
$$

$$
\text{with } C_{am}(S,t) \text{ and } \frac{\partial C_{am}}{\partial S} \text{ continuous}
$$

Just like in *section 5.1* the *Black Scholes Differential Equation* is transformed to the *Heat Equation*, but with one difference. Paul Wilmort et al. in [1] suggests that its best to subtract the payoff from the option value rather than transforming the entire option value. Mathematically the transform will be $C_{am}(S,t) - (S-E) = Ec(x,\tau)$ instead of $C_{am}(S,t) = Ec(x,\tau)$.

$$
\begin{aligned}
&C_{am}(S,t) = Ee^x - E + Ec(x,\tau)\\
&t = T - \frac{\tau}{\sigma^2/2}\\
&S = Ee^x
\end{aligned}
$$

This transformation changes the partial derivatives in the *Black and Scholes Differential Equation* to:

$$
\begin{aligned}
\frac{\partial C_{am}}{\partial t} &= \frac{\partial}{\partial t}\left[S - E + Ec(x,\tau)\right] = -\frac{E\sigma^2}{2}\frac{\partial c}{\partial \tau}\\
\frac{\partial C_{am}}{\partial S} &= \frac{\partial}{\partial S}\left[S - E + Ec(x,\tau)\right] = 1 + \frac{E}{S}\frac{\partial c}{\partial x}\\
\frac{\partial^2 C_{am}}{\partial S^2} &= \frac{\partial}{\partial S}\frac{\partial C_{am}}{\partial S} = \frac{\partial}{\partial S}\left[1 + \frac{E}{S}\frac{\partial c}{\partial x}\right] = \frac{E}{S^2}\left(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}\right)
\end{aligned}
$$

Filling these partial derivatives into the *Black and Scholes differential equation* the following is obtained:

$$
\begin{aligned}
-\frac{E\sigma^2}{2}\frac{\partial c}{\partial \tau} + \frac{1}{2}S^2\sigma^2\frac{E}{S^2}\left(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}\right) + (r - D_0)S\left(1 + \frac{E}{S}\frac{\partial c}{\partial x}\right) - r(Ec(x,\tau) + s - E) = 0\\
-\frac{\partial c}{\partial \tau} + \frac{\partial^2 c}{\partial x^2} + (k' - 1)\frac{\partial c}{\partial x} - kc + f(x) = 0 \qquad (\star)\\
\text{where } k = \frac{r}{\sigma^2/2}\,,\, k' = \frac{r - D_0}{\sigma^2/2} \text{ and } f(x) = (k' - k)e^x + k
\end{aligned}
$$

This transformation changes the *free boundary*, $S_f(t)$, to $x_f(\tau)$. Because of the subtraction of the payoff the conditions on the *free boundary* are greatly relaxed.

$$c(x_f(\tau),\tau) = 0 \qquad\qquad \frac{\partial c(x_f(\tau),\tau)}{\partial x} = 0$$

The payoff of the option changes in the following manner. The time variable t is transformed to τ in the *Heat equation* at time $t = T$, τ is equal to 0. Thus at $\tau = 0$ the *Heat Equation* is at the payoff.

$$C_{am}(S,T) = \begin{cases} S - E \text{ , if } S \geqslant E \\ 0 \qquad \text{ , if } S < E \end{cases} \qquad\qquad c(x,0) = \begin{cases} 0 \qquad \text{ , if } x \geqslant 0 \\ 1 - e^x \text{ , if } x < 0 \end{cases}$$

8.1 Behavior near the Free Boundary

Armed with the just defined transformation of the *Black and Scholes Differential Equation* it is possible to define a differential equation that describes the behavior of the *free boundary* as it moves away from expiry $x_f(0)$. To find this equation we look closer to the $f(x)$ term in $(\star)$. The $f(x)$ term implies the existence of the *optional exercise price*. At expiry $\tau = 0$ if $x \geqslant 0$ the function $c(x,\tau) = \frac{\partial c}{\partial x} = \frac{\partial^2 c}{\partial x^2} = 0$. Filling this into the *Black and Scholes differential equation* only leaves the terms: $\frac{\partial c}{\partial \tau} - f(x) = 0$. If $f(x)$ is negative then $c(x,\tau)$ will become negative if τ increases. This leads to a problem, because this violates the constraint $c(x,\tau) \geqslant MAX\{1 - e^x, 0\}$. This gives the *Free boundary* at $\tau = 0^+$, where 0^+ is the moment just after $\tau = 0$. At this moment the *Free boundary* is at $x_f(0^+) = x_0$, where x_0 is the value such that $f(x_0) = 0$. $f(x) = (k' - k)e^x - k$, thus $x_0 = \ln\left(\frac{k'-k}{k}\right)$. In terms of the *Black and Scholes differential equation* to $S_f(T^-) = \frac{Er}{D_0}$, where T^- is the moment just before time T. This is the value to which the *free boundary* of the option in figure 5 jumps when moving away from T. When $D_0 = 0$ the *free boundary* jumps to infinity at T^-, which is the same result as observed in figure 6. When τ starts increasing $x_f(\tau)$ moves away from x_0 and does not have an exact solution. Willmott et al. approximate near $x = x_0$ and for small values of τ a solution by taking a *Taylor approximation* of $f(x)$ around x_0.

$$f(x) = f(x_0) + \frac{df(x_0)}{dx}(x - x_0) + O((x - x_o)^2)$$

$$f(x_0) = 0 \qquad\qquad \frac{df(x_0)}{dx} = (k' - k)e^{x_0} - k$$

$$f(x) \approx -k(x - x_0)$$

When $c(x,\tau)$ is getting near the *optimal exercise price* the values of $c(x,\tau)$ and $\frac{\partial c}{\partial x}$ should be close to 0. This makes $\frac{\partial^2 c}{\partial x^2}$ higher then $c(x,\tau)$ and $\frac{\partial c}{\partial x}$. Thus the equation that describes the local behavior is defined as:

$$\frac{\partial \hat{c}}{\partial \tau} = \frac{\partial^2 \hat{c}}{\partial x^2} - k(x - x_0) \qquad (\star\star)$$

with free boundary conditions: $\frac{\partial \hat{c}}{\partial x} = \hat{c}(x,\tau) = 0$ at $x = x_f(\tau)$. When $x \to -\infty$, $\frac{\partial^2 \hat{c}}{\partial x^2}$ will go to 0 and the $-k(x - x_0)$ term will behave as $-kx$. Leaving only $\frac{\partial u}{\partial \tau} = -kx$. Integrating this expression shows that $\hat{c}(x,\tau)$ will look like $-kx\tau$ if x goes to $-\infty$. The behavior of $\hat{c}(x,\tau)$ when $x \to -\infty$ will become of great relevance further on.

8.2 Solving for the Free Boundary Using an ODE

The *partial differential equation* ($\star\star$) is solved finding a *similarity solution*, where one reduces a *partial differential equation* into an *ordinary differential equation*. This *similarity solution* is found using the following transformation:

$$\gamma = \frac{x - x_0}{\tau} \qquad\qquad \hat{c}(x,\tau) = \tau^{3/2} c^*(\gamma)$$

The free boundary Wilmott suggests has the form $x_f(\tau) = x_0 + \gamma_0\sqrt{\tau}$

Before executing the transformation on $\hat{c}(x,\tau)$ it is worth noting that the derivative $\frac{\partial\gamma}{\partial\tau} = -\frac{x-x_0}{2\tau^{3/2}} = -\frac{\gamma}{2\tau}$. Using this information this transformation on the partial derivatives in $\hat{c}(x,\tau)$ to:

$$\frac{\partial \hat{c}}{\partial \tau} = \frac{\partial}{\partial \tau}\tau^{3/2}c^*(\gamma) = \frac{3}{2}\sqrt{\tau}c^*(\gamma) + \tau^{3/2}\frac{\partial c^*}{\partial\gamma}\frac{\partial\gamma}{\partial\tau} = \frac{3}{2}\sqrt{\tau}c^*(\gamma) - \sqrt{\tau}\frac{\gamma}{2}\frac{\partial c^*}{\partial\gamma}$$

$$\frac{\partial^2 \hat{c}}{\partial x^2} = \frac{\partial^2}{\partial x^2}\tau^{3/2}c^*(\gamma) = \tau^{3/2}\frac{\partial}{\partial x}\left(\frac{\partial c^*}{\partial\gamma}\frac{1}{\sqrt{\tau}}\right) = \sqrt{\tau}\frac{\partial^2 c^*}{\partial\gamma^2}$$

By filling in these partial derivatives into ($\star\star$) gives an *ordinary differential equation*:

$$\sqrt{\tau}\left(\frac{3}{2}c^*(\gamma) - \frac{\gamma}{2}\frac{dc^*}{d\gamma}\right) = \sqrt{\tau}\frac{d^2c^*}{d\gamma^2} - k(x - x_0)$$

$$\frac{3}{2}c^*(\gamma) - \frac{\gamma}{2}\frac{dc^*}{d\gamma} = \frac{d^2c^*}{d\gamma^2} - k\frac{x - x_0}{\sqrt{\tau}}$$

$$\frac{d^2c^*}{d\gamma^2} + \frac{\gamma}{2}\frac{dc^*}{d\gamma} - \frac{3}{2}c^*(\gamma) = k\gamma \qquad (\star\star\star)$$

This transformation causes the condition on $\hat{c}(x,\tau)$ when x going to $-\infty$ and the *free boundary* conditions to change in the following way:

$$\hat{c}(x,\tau) = \frac{\partial c}{\partial x} = 0 \text{ at } x = x_f(t) \text{ becomes } c^*(\gamma) = \frac{\partial c^*}{\partial\gamma} = 0 \text{ at } \gamma = \gamma_0$$

$$\hat{c}(x,\tau) \to -kx\tau \text{ if } x \to -\infty \text{ becomes } c^*(\gamma) \to -k\gamma \text{ if } \gamma \to -\infty$$

($\star\star\star$) is a *inhomogeneous second-order linear differential equation* which has a solution of the following form: $c^*(\gamma) = c^*_h(\gamma) + c^*_p(\gamma)$. The $c^*_h(\gamma) = Ac^*_{h_1}(\gamma) + Bc^*_{h_2}(\gamma)$ with A and B constants is the *homogeneous solution*. The term $c^*_p(\gamma)$ term is called the *particular solution* which is any solution to the *homogeneous equation* [7]. The *particular solution* is easily found by observing that ($\star\star\star$) is solved by $-k\gamma$. Now for the *homogeneous solution*, $c^*_h(\gamma)$. a solution to $\frac{d^2c^*}{d\gamma^2} + \frac{\gamma}{2}\frac{dc^*}{d\gamma} - \frac{3}{2}c^*(\gamma) = 0$.The first part of the *homogeneous solution* $c^*_{h_1}(\gamma)$ is found by fitting a third order polynomial using *the method of unknown coefficients*:

$$c^*_{h_1}(\gamma) = \beta_0 + \beta_1\gamma + 2\beta_2\gamma^2 + \beta_3\gamma^3$$

$$\frac{dc^*_{h_1}(\gamma)}{d\gamma} = \beta_1 + \beta_2\gamma + 3\beta_3\gamma^2$$

$$\frac{d^2c^*_{h_1}(\gamma)}{d\gamma^2} = 2\beta_2 + 6\beta_3\gamma$$

$$(2\beta_2 + 6\beta_3\gamma) + \frac{\gamma}{2}(\beta_1 + 2\beta_2\gamma + 3\beta_3\gamma^2) - \frac{3}{2}(\beta_0 + \beta_1\gamma + 2\beta_2\gamma^2 + \beta_3\gamma^3) = 0$$

The coefficients $(\beta_0, \beta_1, \beta_2, \beta_3)$ can be found by solving the following system of equations:

$$\begin{aligned}
\frac{3}{2}\beta_3 - \frac{3}{2}\beta_3 = 0 &\Rightarrow \beta_3 = \mathit{free} \\
\beta_2 - \frac{3}{2}\beta_2 = 0 &\Rightarrow \beta_2 = 0 \\
6\beta_3 + \frac{1}{2}\beta_1 - \frac{3}{2}\beta_1 = 0 &\Rightarrow \beta_1 = 6\beta_3 \\
2\beta_2 - \frac{3}{2}\beta_0 = 0 &\Rightarrow \beta_0 = 0
\end{aligned}$$

Since $c^*_{h_1}(\gamma)$ is multiplied with a constant A in the *homogeneous solution* β_3 can be set to 1 without loss.

To find the second part of the *homogeneous solution* the *variation of parameters method* is used. In this method the second part of the *homogeneous solution* is found by $c^*_{h_2}(\gamma) = c^*_{h_1}(\gamma)\alpha(\gamma)$:

$$\begin{aligned}
c^*_{h_2}(\gamma) &= c^*_{h_1}(\gamma)\alpha(\gamma) = \alpha(\gamma)(\gamma^3 + 6\gamma) \\
\frac{dc^*_{h_2}(\gamma)}{d\gamma} &= \alpha(\gamma)(3\gamma^2 + 6) + \frac{d\alpha}{d\gamma}(\gamma^3 + 6\gamma) \\
\frac{d^2 c^*_{h_2}(\gamma)}{d\gamma^2} &= \alpha(\gamma)6\gamma + 2\frac{d\alpha}{d\gamma}(6 + 3\gamma^2) + \frac{d^2\alpha}{d\gamma^2}(6\gamma + \gamma^3)
\end{aligned}$$

Filling this back into the *homogeneous solution* gives:

$$\alpha(\gamma)6\gamma + 2\frac{d\alpha}{d\gamma}(6\gamma + 3\gamma^2) + \frac{d^2\alpha}{d\gamma^2}(6\gamma + \gamma^3) + \frac{\gamma}{2}\left(\alpha(\gamma)(3\gamma^2 + 6) + \frac{d\alpha}{d\gamma}(\gamma^3 + 6\gamma)\right) - \frac{3}{2}\alpha(\gamma^3 + 6\gamma) = 0$$

$$\frac{d\alpha}{d\gamma}\left(12 + 9\gamma^2 + \frac{\gamma^4}{2}\right) + \frac{d^2\alpha}{d\gamma^2}\left(6\gamma + \gamma^3\right) = 0$$

By defining a function $v(\gamma) = \frac{d\alpha}{d\gamma}$ one can solve for $\alpha(\gamma)$ using the *separation of variables method*.

$$\begin{aligned}
v(\gamma)\left(12 + 9\gamma^2 + \frac{\gamma^4}{2}\right) + \frac{dv}{d\gamma}\left(6\gamma + \gamma^3\right) &= 0 \\
\frac{dv}{v(\gamma)} &= \frac{12 + 9\gamma^2 + \frac{\gamma^4}{2}}{6\gamma + \gamma^3} d\gamma \\
\int \frac{dv}{v(\gamma)} &= \int \frac{12 + 9\gamma^2 + \frac{\gamma^4}{2}}{6\gamma + \gamma^3} d\gamma \\
\ln|v(\gamma)| &= -\int \frac{\gamma}{2} d\gamma - \int \frac{6\gamma^2}{\gamma^3 + 6\gamma} d\gamma - \int \frac{12}{\gamma^3 + 6\gamma} d\gamma \\
\ln|v(\gamma)| &= -\frac{\gamma^2}{4} - 2\ln|\gamma| - 2\ln|\gamma^3 + 6\gamma| + \omega \\
v(\gamma) &= e^{-\frac{\gamma^2}{4}} \frac{\omega^*}{\gamma^2(\gamma^2 + 6)^2} \qquad \text{with } \omega^* = e^{\omega}
\end{aligned}$$

To avoid confusion ω denotes the integrating constant instead of the usual c. By integrating $v(\gamma)$ the function $\alpha(\gamma)$ is found:

$$\begin{aligned}\frac{\alpha(\gamma)}{\omega^*} &= \int e^{-\frac{\gamma^2}{4}} \frac{1}{\gamma^2(\gamma^2+6)^2} d\gamma \\ &= \underbrace{\int \frac{e^{-\frac{\gamma^2}{4}}}{36\gamma^2} d\gamma}_{I_1} - \underbrace{\int \frac{(\gamma^2+12)e^{-\frac{\gamma^2}{4}}}{36(\gamma^2+6)^2} d\gamma}_{I_2}\end{aligned}$$

The integrals I_1 and I_2 are calculated separately.

$$\begin{aligned}I_1 &= \int \frac{e^{-\frac{\gamma^2}{4}}}{36\gamma^2} d\gamma = \frac{1}{36} \int e^{-\frac{\gamma^2}{4}} \left(\frac{d}{d\gamma} - \frac{1}{\gamma} \right) d\gamma \\ &= \frac{1}{36} \left(-\frac{e^{-\frac{\gamma^2}{4}}}{\gamma} - \frac{1}{2} \int_a^\gamma e^{-\frac{s^2}{4}} ds \right) \\ &= -\frac{1}{36} \frac{e^{-\frac{\gamma^2}{4}}}{\gamma} - \frac{1}{72} \int_a^\gamma e^{-\frac{s^2}{4}} ds\end{aligned}$$

Before calculating I_2 the integral of $\frac{\gamma^2+12}{36(\gamma^2+6)^2}$ needs to be found:

$$\int \frac{\gamma^2+12}{36(\gamma^2+6)^2} d\gamma = \frac{1}{36} \left(\int \frac{\gamma^2}{(\gamma^2+6)^2} d\gamma + \int \frac{12}{(\gamma^2+6)^2} d\gamma \right)$$

These two separate integrals can be solved by an *inverse trigonometric substitution* $\theta = \tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right)$ [7].

$$\int \frac{\gamma^2+12}{36(\gamma^2+6)^2} d\gamma = \frac{1}{144} \left(\frac{2\gamma}{\gamma^2+6} + \sqrt{6} \tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) \right)$$

Knowing this integral makes it possible to use the *product rule* for I_2:

$$\begin{aligned}I_2 &= \int \frac{(\gamma^2+12)e^{-\frac{\gamma^2}{4}}}{36(\gamma^2+6)^2} d\gamma = \int e^{-\frac{\gamma^2}{4}} \left(\frac{d}{d\gamma} \frac{1}{144} \left(\frac{2\gamma}{\gamma^2+6} + \sqrt{6} \tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) \right) \right) d\gamma \\ &= \frac{1}{144} \left(\frac{2\gamma}{\gamma^2+6} + \sqrt{6} \tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) \right) e^{-\frac{\gamma^2}{4}} - \int \frac{1}{144} \left(\frac{2\gamma}{\gamma^2+6} + \sqrt{6} \tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) \right) \left(\frac{-\gamma}{2} \right) e^{-\frac{\gamma^2}{4}} d\gamma \\ &= \frac{1}{144} \left(\frac{2\gamma}{\gamma^2+6} + \sqrt{6} \tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) \right) e^{-\frac{\gamma^2}{4}} + \frac{1}{288} \left(\underbrace{\int \frac{2\gamma^2 e^{-\frac{\gamma^2}{4}}}{\gamma^2+6} d\gamma}_{I_3} + \int \sqrt{6} \tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) \gamma e^{-\frac{\gamma^2}{4}} d\gamma \right)\end{aligned}$$

Now the integral I_3 is solved:

$$\begin{aligned} I_3 &= 2\int \frac{\gamma^2 e^{-\frac{\gamma^2}{4}}}{\gamma^2+6} d\gamma \\ &= 2\int \left(\frac{d}{d\gamma}\left(\gamma - \sqrt{6}\tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right)\right)\right) e^{-\frac{\gamma^2}{4}} d\gamma \\ &= 2\left(\left(\gamma - \sqrt{6}\tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right)\right) e^{-\frac{\gamma^2}{4}} + \underbrace{\int \frac{\gamma^2}{2} e^{-\frac{\gamma^2}{4}} d\gamma}_{I_4} - \int \frac{\gamma}{2}\sqrt{6}\tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) e^{-\frac{\gamma^2}{4}} d\gamma\right) \end{aligned}$$

Integral I_4 is solved by:

$$I_4 = \int \frac{\gamma^2}{2} e^{-\frac{\gamma^2}{4}} d\gamma = \int_a^\gamma e^{-\frac{s^2}{4}} ds - e^{-\frac{\gamma^2}{4}}\gamma$$

Now I_4 is filled back into I_3:

$$I_3 = 2\left(\left(\gamma - \sqrt{6}\tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right)\right) e^{-\frac{\gamma^2}{4}} + \int_a^\gamma e^{-\frac{s^2}{4}} ds - e^{-\frac{\gamma^2}{4}}\gamma - \int \frac{\gamma}{2}\sqrt{6}\tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) e^{-\frac{\gamma^2}{4}} d\gamma\right)$$

Now I_3 is filled back into I_2:

$$\begin{aligned} I_2 = &\frac{1}{144}\left(\frac{2\gamma}{\gamma^2+6} + \sqrt{6}\tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right)\right) e^{-\frac{\gamma^2}{4}} + \frac{1}{288}\left(2\left(\gamma - \sqrt{6}\tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right)\right) e^{-\frac{\gamma^2}{4}} + \int_a^\gamma e^{-\frac{s^2}{4}} ds\right) \\ &+ \frac{2}{288}\left(-\gamma e^{-\frac{\gamma^2}{4}} - \int \frac{\gamma}{2}\sqrt{6}\tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) e^{-\frac{\gamma^2}{4}} d\gamma\right) + \frac{1}{288}\int \sqrt{6}\tan^{-1}\left(\frac{\gamma}{\sqrt{6}}\right) \gamma e^{-\frac{\gamma^2}{4}} d\gamma \\ = &\frac{1}{72}\left(\frac{\gamma e^{-\frac{\gamma^2}{4}}}{\gamma^2+6} + \frac{1}{2}\int_a^\gamma e^{-\frac{s^2}{4}} ds\right) \end{aligned}$$

Filling this back into $\frac{\alpha(\gamma)}{\omega^*}$ gives:

$$\begin{aligned} \frac{\alpha(\gamma)}{\omega^*} &= \underbrace{\int \frac{e^{-\frac{\gamma^2}{4}}}{36\gamma^2} d\gamma}_{I_1} - \underbrace{\int \frac{(\gamma^2+12)e^{-\frac{\gamma^2}{4}}}{36(\gamma^2+6)^2} d\gamma}_{I_2} \\ &= -\frac{1}{36}\frac{e^{-\frac{\gamma^2}{4}}}{\gamma} - \frac{1}{72}\int_a^\gamma e^{-\frac{s^2}{4}} ds - \frac{1}{72}\left(\frac{\gamma e^{-\frac{\gamma^2}{4}}}{\gamma^2+6} + \frac{1}{2}\int_a^\gamma e^{-\frac{s^2}{4}} ds\right) \\ \alpha(\gamma) &= \left(\left(-\frac{1}{36\gamma} - \frac{\gamma}{72(\gamma^2+6)}\right) e^{-\frac{\gamma^2}{4}} - \frac{1}{48}\int_a^\gamma e^{-\frac{s^2}{4}} ds\right)\omega^* \end{aligned}$$

Because the *homogeneous solution* has the following form $c_h^*(\gamma) = Ac_{h_1}^*(\gamma) + Bc_{h_2}^*(\gamma)$ and $c_{h_2}^*(\gamma) = c_{h_1}^*(\gamma)\alpha(\gamma)$ the constant ω^* will be absorbed in the constant B, thus without loss ω^* can be set to 1.

$$
\begin{aligned}
c^*_{h_2}(\gamma) &= c^*_{h_1}(\gamma)\alpha(\gamma) \\
&= \left(\gamma^3+6\gamma\right)\left(\left(-\frac{1}{36\gamma}-\frac{\gamma}{72(\gamma^2+6)}\right)e^{-\frac{\gamma^2}{4}}-\frac{1}{48}\int_a^\gamma e^{-\frac{s^2}{4}}ds\right) \\
&= -\left(\frac{\gamma^3+6\gamma}{36\gamma}+\frac{\gamma^4+6\gamma^2}{72(\gamma^2+6)}\right)e^{-\frac{\gamma^2}{4}}-\frac{\gamma^3+6\gamma}{48}\int_a^\gamma e^{-\frac{s^2}{4}}ds \\
&= \frac{1}{12}\left(-\left(\frac{\gamma^2}{3}+\frac{6}{3}+\frac{\gamma^2}{6}\right)e^{-\frac{\gamma^2}{4}}-\frac{1}{4}\int_a^\gamma e^{-\frac{s^2}{4}}ds\right) \\
&= \frac{1}{12}\left(-\left(\frac{\gamma^2}{2}+2\right)e^{-\frac{\gamma^2}{4}}-\frac{\gamma^3+6\gamma}{4}\int_a^\gamma e^{-\frac{s^2}{4}}ds\right) \\
&= -\frac{1}{6}\left(\left(\gamma^2+4\right)e^{-\frac{\gamma^2}{4}}+\frac{\gamma^3+6\gamma}{2}\int_a^\gamma e^{-\frac{s^2}{4}}ds\right)
\end{aligned}
$$

First, the factor $-\frac{1}{6}$ term is moved to the constant B. Secondly, the lower limit of the integral $\int_a^\gamma e^{-\frac{s^2}{4}}ds$ will only add a multiple of $c^*_{h_1}$, thus without loss a can be set to $-\infty$. By incorporating these observations into the *second homogeneous solution* we obtain:

$$
c^*_{h_2}(\gamma) = \left(\gamma^2+4\right)e^{-\frac{\gamma^2}{4}}+\frac{\gamma^3+6\gamma}{2}\int_{-\infty}^\gamma e^{-\frac{s^2}{4}}ds
$$

Now both the *homogeneous solutions* and the *particular solution* are found it's possible to fill in $(\star\star\star)$.

$$
c^*(\gamma) = -k\gamma + A(\gamma^3+6\gamma) + B\left(\left(\gamma^2+4\right)e^{-\frac{\gamma^2}{4}}+\frac{\gamma^3+6\gamma}{2}\int_{-\infty}^\gamma e^{-\frac{s^2}{4}}ds\right)
$$

Now the condition $c^*(\gamma) \to -k\gamma$ as $\gamma \to -\infty$ is applied. For this condition to hold A has to be 0, because $(\gamma^3+6\gamma)$ will go to $-\infty$ if γ goes to $-\infty$.

$$
c^*(\gamma) = -k\gamma + B\left(\left(\gamma^2+4\right)e^{-\frac{\gamma^2}{4}}+\frac{\gamma^3+6\gamma}{2}\int_{-\infty}^\gamma e^{-\frac{s^2}{4}}ds\right)
$$

The part of the equation multiplied by B will remain, because it will go to 0 leaving only $-k\gamma$, which satisfies the condition.

The *free boundary* has the following form $x_f(\tau) = x_0 + \gamma_0\sqrt{\tau}$. All what is left to do is to find γ_0 using the *free boundary* conditions of $c^*(\gamma)$:

$$
\begin{aligned}
c^*(\gamma) &= \frac{dc^*(\gamma)}{d\gamma} = 0 \text{ at } \gamma = \gamma_0 \\
-k\gamma_0 + Bc^*(\gamma_0) &= -k + B\frac{dc^*_{h_2}(\gamma_0)}{d\gamma} \\
-\gamma_0 + B^*c^*(\gamma_0) &= -1 + B^*\frac{dc^*_{h_2}(\gamma_0)}{d\gamma} \qquad \text{with } B^* = B/k \\
B^*c^*_{h_2}(\gamma_0) &= \gamma_0 \ \&\ B^*\frac{dc^*_{h_2}(\gamma_0)}{d\gamma} = 1
\end{aligned}
$$

This leaves two equations with two unknowns: γ_0 and B^*. By taking $B^* = \frac{\gamma_0}{c^*_{h_2}(\gamma_0)}$ an equation in terms of γ_0 is obtained:

$$\gamma_0 \frac{dc^*_{h_2}(\gamma_0)}{d\gamma} = c^*_{h_2}(\gamma_0)$$

The derivative $\frac{dc^*_{h_2}(\gamma)}{d\gamma}$ at γ_0 is:

$$\begin{aligned}
\frac{dc^*_{h_2}(\gamma_0)}{d\gamma} &= \frac{d}{d\gamma}\left(\left(\gamma^2+4\right)e^{-\frac{\gamma^2}{4}} + \frac{\gamma^3+6\gamma}{2}\int_{-\infty}^{\gamma} e^{-\frac{s^2}{4}}\,ds\right)\Big|_{\gamma=\gamma_0} \\
&= 3\gamma_0 e^{-\frac{\gamma^2}{4}} + \frac{3\gamma_0^2+6}{2}\int_{-\infty}^{\gamma_0} e^{-\frac{s^2}{4}}\,ds
\end{aligned}$$

Solving for γ_0:

$$\begin{aligned}
\gamma_0\left(3\gamma_0 e^{-\frac{\gamma^2}{4}} + \frac{3\gamma_0^2+6}{2}\int_{-\infty}^{\gamma_0} e^{-\frac{s^2}{4}}\,ds\right) &= \left(\left(\gamma_0^2+4\right)e^{-\frac{\gamma_0^2}{4}} + \frac{\gamma_0^3+6\gamma_0}{2}\int_{-\infty}^{\gamma_0} e^{-\frac{s^2}{4}}\,ds\right) \\
2\gamma_0^2 e^{-\frac{\gamma_0^2}{4}} - 4e^{-\frac{\gamma_0^2}{4}} &= 2\gamma_0^3\int_{-\infty}^{\gamma_0} e^{-\frac{s^2}{4}}\,ds \\
2(2-\gamma_0) &= \gamma_0^3 e^{-\frac{\gamma_0^2}{4}}\int_{-\infty}^{\gamma_0} e^{-\frac{s^2}{4}}\,ds
\end{aligned}$$

Which can be solved using symbolic toolbox of *Matlab*:

```
syms x
req = (x^3) * exp((x^2)/4) * 2 * (pi^(1/2)) * normcdf(x, 0, (2^(1/2))) == 2 * (2 - x^2);
solve(req)
x = 0.90344659788434394725487986458608
```

Now the only thing left to do is transforming the *free boundary* back in financial variables.

$$\begin{aligned}
x_f(\tau) &= x_0 + \gamma_0\sqrt{\tau} && \text{as } \tau \to 0 \\
S_f(t) &= \frac{rE}{D}\left(1 + \gamma_0\sqrt{\frac{\sigma^2}{2}(T-t)}\right) && \text{as } t \to T
\end{aligned}$$

8.3 Comparison with the Finite-Difference Method

In this section the *free boundary* obtained in *section 7* is compared with the asymptotic solution $S_f(t) = \frac{rE}{D_0}\left(1 + \gamma_0\sqrt{\frac{\sigma^2}{2}(T-t)}\right)$. These two solutions are compared by looking at the FX option that is used throughout this paper. In figure 8 the *free boundary* from figure 5 and the asymptotic solution are shown.

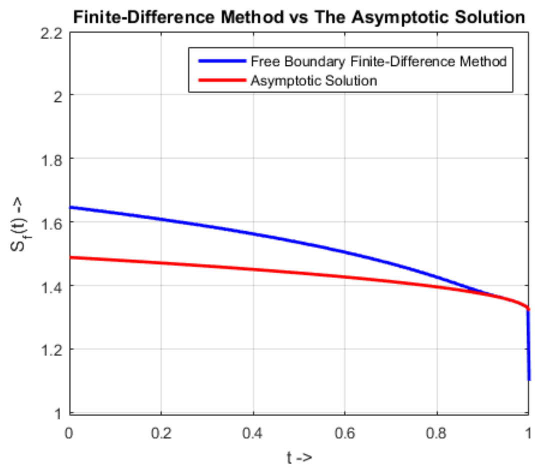
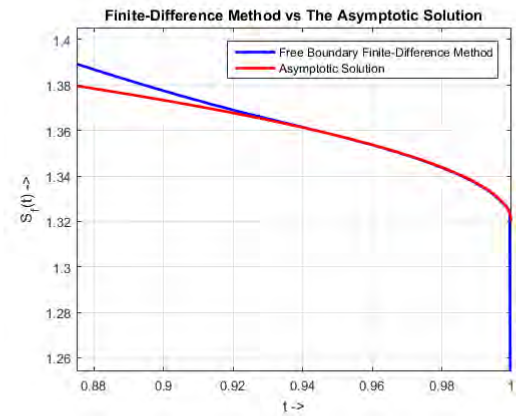
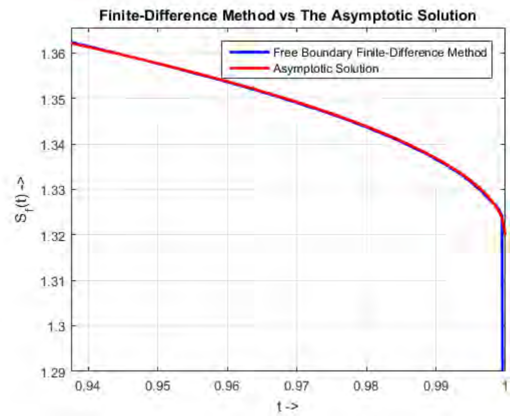


Figure 8

Initially it is observed that both solutions are relatively close to each other and after a certain point they move away from each other. This is because the asymptotic solution is only correct around expiry for small changes in S and t, thus for $t << T$ it will not be correct. Secondly it is observed that the asymptotic solution does not display the same drop in option price when it reaches T. This is because the asymptotic solution describes the behavior of the *free boundary* when $t \rightarrow T$. When T is plugged in it describes the value of the option right before at T^-. By definition the option value at expiry is $MAX\{S-E, 0\}$. When observing the *free boundary* obtained by the *finite-difference method* is moving away from expiry it jumps up it comes very close to this value. Due to the discretization used in obtaining the *finite-difference method* in *section 6* it is not possible to correctly evaluate an expression like T^- (see figure 9). However this asymptotic solution does describe the previously unexplained drop in the *free boundary* right before expiry: Just before the *free boundary* goes to E at time T it is at $\frac{Er}{D_0}$.



(a) Figure 8 zoomed into time interval [0.88,1]



(b) Figure 8 zoomed into time interval [0.94,1]

Figure 9

9 Conclusion/Discussion

For an American option on an asset with one dividend-payment during the lifetime of an option it is optimal to exercise it either right before the dividend payment or at expiry, For the American Option on an asset with a continuous dividend yield there exists an optimal exercise price during the entire lifetime of the contract. The asymptotic solution to the free boundary posed in this paper is only valid for assets with a dividend yield strictly lower then the risk-free rate. Due to the limited time for writing this paper the asymptotic solution for assets with an higher dividend yield is not derived. Further Research will be required for finding this solution.

Appendix I: Ito's Lemma

Ito's Lemma

Let a stochastic process $dG = A(G,t)dG + B(G,t)dt$ with dG a *Wiener Proses*. Then the dynamics of a function of G and t, $f(G,t)$, are given by:

$$df(G,t) = \frac{\partial f(G,t)}{\partial G} A(G,t)\, dX + \left(\frac{\partial f(G,t)}{\partial G} B(G,t) + \frac{\partial f(G,t)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(G,t)}{\partial G^2} A(G,t)^2 \right) dt$$

proof:
We look at a small change in $df(G,t) = f(G+dG, t+dt) - f(G,t)$. With *Taylors Theorem* we get:

$$\begin{aligned} f(G+dG, t+dt) = f(G,t) &+ \frac{\partial f(G,t)}{\partial G}(G+dG-G) + \frac{\partial f(G,t)}{\partial t}(t+dt-t) \\ &+ \frac{1}{2}\frac{\partial^2 f(G,t)}{\partial G^2}(G+dG-G)^2 + \frac{\partial^2 f(G,t)}{\partial t^2}(t+dt-t)^2 \\ &+ \frac{\partial^2 f(G,t)}{\partial G \partial t}(t+dt-t)(G+dG-G) + \dots \end{aligned}$$

Filling this back into $df(G,t) = f(G+dG, t+dt) - f(G,t)$ we get:

$$\begin{aligned} df(G,t) = f(G,t) &+ \frac{\partial f(G,t)}{\partial G} dG + \frac{\partial f(G,t)}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f(G,t)}{\partial G^2} dG^2 + \frac{\partial^2 f(G,t)}{\partial t^2} dt^2 \\ &+ \frac{\partial^2 f(G,t)}{\partial g \partial t} dt dG + \dots - f(G,t) \end{aligned}$$

Now we fill in $dG = A(G,t)\, dX + B(G,t)\, dt$ into the *Taylor Expansion*:

$$\begin{aligned} df(G,t) = f(G,t) &+ \frac{\partial f(G,t)}{\partial G}(A(G,t)\, dX + B(G,t)\, dt) + \frac{\partial f(G,t)}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f(G,t)}{\partial G^2}(A(G,t)\, dX + B(G,t)\, dt)^2 \\ &+ \frac{\partial^2 f(G,t)}{\partial t^2} dt^2 + \frac{\partial^2 f(G,t)}{\partial g \partial t} dt (A(G,t)\, dX + B(G,t)\, dt) + \dots - f(G,t) \\ df(G,t) = &\frac{\partial f(G,t)}{\partial G} A(G,t)\, dX + \frac{\partial f(G,t)}{\partial G} B(G,t)\, dt + \frac{\partial f(G,t)}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f(G,t)}{\partial G^2} A(G,t)^2\, dX^2 \\ &+ \frac{\partial^2 f(G,t)}{\partial G^2} B(G,t)\, A(G,t)\, dt dX + \frac{1}{2}\frac{\partial^2 f(G,t)}{\partial G^2} B(G,t)\, dt^2 + \frac{\partial^2 f(G,t)}{\partial t^2} dt^2 \\ &+ \frac{\partial^2 f(G,t)}{\partial g \partial t} A(G,t)\, dX dt + \frac{\partial^2 f(G,t)}{\partial g \partial t} B(G,t)\, dt^2 + \dots \end{aligned}$$

Now we let dt go to 0. When $dt -> 0$ $dX^2 -> dt$, thus all terms grater then dt fall off:

$$\begin{aligned} df(G,t) &= \frac{\partial f(G,t)}{\partial G} A(G,t)\, dX + \frac{\partial f(G,t)}{\partial G} B(G,t)\, dt + \frac{\partial f(G,t)}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f(G,t)}{\partial G^2} A(G,t)^2\, dt \\ df(G,t) &= \frac{\partial f(G,t)}{\partial G} A(G,t)\, dX + \left(\frac{\partial f(G,t)}{\partial G} B(G,t) + \frac{\partial f(G,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(G,t)}{\partial G^2} A(G,t)^2 \right) dt \end{aligned}$$

Appendix II: European Call Option on an asset with a constant dividend yield

European Call Option on an asset with a constant dividend yield

The dynamics of an asset with a constant dividend yield are given by: $dS = \sigma S dX + S(\mu - D_0)dt$. The *Black and Scholes differential equation* with an asset like this as underlying is given by:

$$\frac{\partial C_{eu}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2 + (r - D_0)S\frac{\partial C_{eu}}{\partial S} - rC_{eu} = 0$$

$C_{eu}(S,t)$ denotes the price of the call option. Solving the differential equation for $C_{eu}(S,t)$ gives:

$$C_{eu}(S,t) = Se^{-D_0(T-t)}\mathcal{N}(d_{10}) - Ee^{-r(T-t)}\mathcal{N}(d_{20})$$

$$\text{with } d_{10} = \frac{\ln(S/E) + (r - D_0 + \frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{T-t}} \text{ and } d_{20} = \frac{\ln(S/E) + (r - D_0 - \frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{T-t}}$$

The value of a European call option is denoted as $C_{eu}(S,t)$. $C_{eu}(S,t)$ is calculated by plugging $dS = \sigma S dX + S(\mu - D_0)dt$ into *Ito's lemma*:

$$dC_{eu}(S,t) = \frac{\partial C_{eu}}{\partial S}\sigma S dX + \left(\frac{\partial C_{eu}}{\partial S}S(\mu - D_0) + \frac{\partial C_{eu}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2\right)dt$$

Now we make a replicating portfolio consisting of Δ stocks and π zero coupon bonds. This portfolio replicates the value of the option, such that we always have: $C_{eu} = \Delta S + \pi$. The dynamics of holding this replicating portfolio are: $d\pi + \Delta dS + D_0\Delta S dt$, thus we get $dC_{eu} = d\pi + \Delta dS + D_0 \Delta S dt$. Filling this into our expression from the call option gives:

$$\begin{aligned}
d\pi + \Delta dS + D_0\Delta S dt &= \frac{\partial C_{eu}}{\partial S}\sigma S dX + \left(\frac{\partial C_{eu}}{\partial S}S(\mu - D_0) + \frac{\partial C_{eu}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2\right)dt \\
d\pi &= \frac{\partial C_{eu}}{\partial S}\sigma S dX + \left(\frac{\partial C_{eu}}{\partial S}S(\mu - D_0) + \frac{\partial C_{eu}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2\right)dt - \Delta dS - D_0\Delta S dt \\
d\pi &= \frac{\partial C_{eu}}{\partial S}\sigma S dX + \left(\frac{\partial C_{eu})}{\partial S}S(\mu - D_0) + \frac{\partial C_{eu}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2\right)dt \\
&\quad - \Delta(\sigma S dX + S(\mu - D_0)dt) - D_0\Delta S dt
\end{aligned}$$

The goal of this replicating portfolio is to eliminate the randomness of the option. The only term that is causing this equation to be random is dX. To eliminate the dX we take $\Delta = \frac{\partial C_{eu}}{\partial S}$ in stocks. This will make both terms involving dX cancel:

$$\begin{aligned}
d\pi &= \frac{\partial C_{eu}}{\partial S}\sigma S dX + \left(\frac{\partial C_{eu})}{\partial S}S(\mu - D_0) + \frac{\partial C_{eu}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2\right)dt \\
&\quad - \frac{\partial C_{eu}}{\partial S}\sigma S dX - \frac{\partial C_{eu}}{\partial S}S(\mu - D_0)dt - D_0\frac{\partial C_{eu}}{\partial S}S dt \\
d\pi &= \frac{\partial C_{eu}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2 dt - D_0\frac{\partial C_{eu}}{\partial S}S dt
\end{aligned}$$

Now we use the fact that a *zero coupon bond* has the following value: $\pi(t) = Ke^{-r(T-t)}$ with K the current value of the bond and r the risk free rate rate. This is the result of a separable differential equation: $d\pi = \pi r dt$.

$$\begin{aligned}
\pi r dt &= \frac{\partial C_{eu}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2 dt - D_0\frac{\partial C_{eu}}{\partial S}Sdt \\
(C_{eu} - S\Delta)rdt &= \frac{\partial C_{eu}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2 dt - D_0\frac{\partial C_{eu}}{\partial S}Sdt \\
(C_{eu} - \frac{\partial C_{eu}}{\partial S}S)rdt &= \frac{\partial C_{eu}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2 dt - D_0\frac{\partial C_{eu}}{\partial S}Sdt \\
(C_{eu} - \frac{\partial C_{eu}}{\partial S}S)r &= \frac{\partial C_{eu}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2 - D_0\frac{\partial C_{eu}}{\partial S}S
\end{aligned}$$

By rearranging the terms we obtain the *Black and Scholes Differential Equation* for a dividend paying asset.

$$\frac{\partial C_{eu}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{eu}}{\partial S^2}\sigma^2 S^2 + (r - D_0)S\frac{\partial C_{eu}}{\partial S} - rC_{eu} = 0$$

To find a solution to this equation it is transformed to the *Heat Equation* using the following variable transformation:

$$\begin{aligned}
&C_{eu}(S,t) = Ev(x,\tau) \\
&t = T - \frac{\tau}{\sigma^2/2} \\
&S = Ee^x
\end{aligned}$$

This transformation changes the partial derivatives in the *Black and Scholes Differential Equation* to:

$$\begin{aligned}
\frac{\partial C_{eu}}{\partial t} &= E\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = E\frac{\partial v}{\partial \tau}\frac{\partial}{\partial t}(T-t)\frac{\sigma^2}{2} = -\frac{E\sigma^2}{2}\frac{\partial v}{\partial \tau} \\
\frac{\partial C_{eu}}{\partial S} &= E\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = E\frac{\partial v}{\partial x}\frac{\partial}{\partial S}ln\left(\frac{S}{E}\right) = \frac{E}{S}\frac{\partial v}{\partial x} \\
\frac{\partial^2 C_{eu}}{\partial S^2} &= \frac{\partial}{\partial S}\frac{\partial C_{eu}}{\partial S} = \frac{\partial}{\partial S}\frac{E}{S}\frac{\partial v}{\partial x} = \frac{E}{S^2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right)
\end{aligned}$$

Filling these transformed partial derivatives into the *Black and Scholes Differential Equation* gives:

$$-\frac{E\sigma^2}{2}\frac{\partial v}{\partial \tau} + \frac{1}{2}S^2\sigma^2\frac{E}{S^2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + (r - D_0)S\frac{E}{S}\frac{\partial v}{\partial x} - Ev(x,\tau) = 0$$

$$\frac{\partial v}{\partial \tau} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}(k' - 1) - kv(x,\tau) = 0$$

$$\text{where} \qquad k' = \frac{r - D_0}{\sigma^2/2} \qquad \text{and} \qquad k = \frac{r}{\sigma^2/2}$$

Wat is left is a differential equation with constant coefficients. With the transformation $v(x,\tau) = u(x,\tau)e^{x\alpha + \tau\beta}$ we obtain the partial derivative:

$$\frac{\partial v}{\partial \tau} = \beta e^{x\alpha+\tau\beta} u(x,\tau) + \frac{\partial u}{\partial \tau} e^{x\alpha+\tau\beta}$$
$$\frac{\partial v}{\partial x} = \alpha e^{x\alpha+\tau\beta} u(x,\tau) + \frac{\partial u}{\partial x} e^{x\alpha+\tau\beta}$$
$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} e^{x\alpha+\tau\beta} + 2\alpha \frac{\partial u}{\partial x} e^{x\alpha+\tau\beta} + \alpha^2 u(x,\tau) e^{x\alpha+\tau\beta}$$

filling this into $\frac{\partial v}{\partial \tau} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}(k'-1) - kv(x,\tau) = 0$ and dividing out all $e^{x\alpha+\tau\beta}$ terms gives:

$$-\beta u(x,\tau) - \frac{\partial u}{\partial \tau} + \alpha^2 u(x,\tau) + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k'-1)\left(\alpha u(x,\tau) + \frac{\partial u}{\partial x}\right) - ku(x,\tau) = 0$$
$$\text{by setting } \alpha = -\frac{1}{2}(k'-1) \text{ and } \beta = -\frac{1}{4}(k'-1)^2 - k$$
$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ this equation is the single dimensional *Heat equation* which has a fundamental solution: $\frac{1}{2\sqrt{\tau\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-\frac{(s-x)^2}{2\tau}} ds$. The starting point of the *Heat equation* is found by looking at the transformation done on the time variable t in the *Black and Scholes differential equation*. When the transform to the *Heat equation* was preformed t was set to $T - \tau / \frac{\sigma^2}{2}$ implying that when $\tau = 0$ t is equal to T. At time T the European call option is at expiry, thus the value of the option is the payoff: $MAX\{S-E, 0\}$.

$$C_{eu}(S,T) = MAX\{S-E,0\}$$
$$Ev(x,0) = MAX\{Ee^x - E, 0\}$$
$$Eu(x,0)\, e^{-x\frac{k'-1}{2} - 0\left(\frac{(k'-1)^2}{4}+k\right)} = MAX\{Ee^x - E, 0\}$$
$$u(x,0) = MAX\{e^{x\frac{k'+1}{2}} - e^{x\frac{k'-1}{2}}, 0\}$$

With the fundamental solution to the *Heat equation* and the initial condition $u(x,0)$ we can setup the solution:

$$u(x,\tau) = \frac{1}{4\sqrt{\tau\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-\frac{(s-x)^2}{2\tau}} ds$$
$$z = \frac{s-x}{\sqrt{2\tau}} \qquad dz = \frac{ds}{\sqrt{2\tau}}$$
$$u(x,\tau) = \frac{\sqrt{2\tau}}{2\sqrt{\tau\pi}} \int_{-\infty}^{\infty} u(z\sqrt{2\tau}+x, 0) e^{-\frac{z^2}{2}} dz$$
$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} MAX\{e^{\left(z\sqrt{2\tau}+x\right)\frac{k'+1}{2}} - e^{\left(z\sqrt{2\tau}+x\right)\frac{k'-1}{2}}, 0\} e^{-\frac{z^2}{2}} dz$$
$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\left(z\sqrt{2\tau}+x\right)\frac{k'+1}{2}} - e^{\left(z\sqrt{2\tau}+x\right)\frac{k'-1}{2}} e^{-\frac{z^2}{2}} dz$$
$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\left(z\sqrt{2\tau}+x\right)\frac{k'+1}{2}} e^{\frac{-z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\left(z\sqrt{2\tau}+x\right)\frac{k'-1}{2}} e^{\frac{-z^2}{2}} dz$$

Only the first integral will be worked out, because the second integral is very similar to the first one.

$$\frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\left(z\sqrt{2\tau}+x\right)\frac{k'+1}{2}} e^{-\frac{z^2}{2}}dz = \frac{e^{x\frac{k'+1}{2}+\frac{\tau}{4}(k'+1)^2}}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\pi}}}^{\infty} e^{-\frac{1}{2}\left(z-\frac{1}{2}(k'+1)\sqrt{2\tau}\right)^2}dz$$

$$\rho = z-\frac{1}{2}(k'+1)\sqrt{2\tau} \qquad d\rho = dz$$

$$= \frac{e^{x\frac{k'+1}{2}+\frac{\tau}{4}(k'+1)^2}}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}-\frac{1}{2}(k'+1)\sqrt{2\tau}}^{\infty} e^{\frac{-\rho^2}{2}}d\rho$$

$$= e^{x\frac{k'+1}{2}+\frac{\tau}{4}(k'+1)^2}\mathcal{N}\left(\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k'+1)\sqrt{2\tau}\right)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\left(z\sqrt{2\tau}+x\right)\frac{k'-1}{2}} e^{-\frac{z^2}{2}}dz = e^{x\frac{k'-1}{2}+\frac{\tau}{4}(k'-1)^2}\mathcal{N}\left(\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k'-1)\sqrt{2\tau}\right)$$

To be consistent with the notation of the book *The Mathematics of Financial Derivatives* the quantity $\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k'+1)$ will be called d_{10} and $\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k'-1)$ d_{20}.

$$u(x,\tau) = e^{x\frac{k'+1}{2}+\frac{\tau}{4}(k'+1)^2}\mathcal{N}(d_{10}) - e^{x\frac{k'-1}{2}+\frac{\tau}{4}(k'-1)^2}\mathcal{N}(d_{20})$$

$$v(x,\tau)\, e^{x\frac{k'-1}{2}+\tau\left(\frac{(k'-1)^2}{4}+k\right)} = e^{x\frac{k'+1}{2}+\frac{\tau}{4}(k'+1)^2}\mathcal{N}(d_{10}) - e^{x\frac{k'-1}{2}+\frac{\tau}{4}(k'-1)^2}\mathcal{N}(d_{20})$$

$$v(x,\tau) = e^{x+k'\tau-k\tau}\mathcal{N}(d_{10}) = e^{-k\tau}\mathcal{N}(d_{20})$$

$$\text{filling } k' = \frac{r-D_0}{\sigma^2/2} \text{ and } k = \frac{r}{\sigma^2/2} \text{ back in}$$

$$v(x,\tau) = e^{x+\frac{r-D_0}{\sigma^2/2}\tau-\frac{r}{\sigma^2/2}\tau}\mathcal{N}(d_{10}) = e^{-\frac{r}{\sigma^2/2}\tau}\mathcal{N}(d_{20})$$

$$\text{filling } v(x,\tau) = \frac{C_{eu}(S,t)}{E}\text{ , } \tau = (T-t)\frac{\sigma^2}{2} \text{ and } x = \ln(\frac{S}{E}) \text{ back in}$$

$$\frac{C_{eu}(S,t)}{E} = \frac{S}{E}e^{-D_0(T-t)\frac{\frac{\sigma^2}{2}}{\sigma^2/2}}\mathcal{N}(d_{10}) - e^{-r(T-t)\frac{\frac{\sigma^2}{2}}{\sigma^2/2}}\mathcal{N}(d_{10})$$

Working the last expression out and transforming d_{10} and d_{20} gives the value of an European call option on a continuous dividend paying asset:

$$C_{eu}(S,t) = Se^{-D_0(T-t)}\mathcal{N}(d_{10}) - Ee^{-r(T-t)}\mathcal{N}(d_{20})$$

$$\text{with } d_{10} = \frac{\ln(S/E)+(r-D_0+\frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{T-t}} \text{ and } d_{20} = \frac{\ln(S/E)+(r-D_0-\frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{T-t}}$$

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