Managing Financial Risks

The Application of Extreme Value Theory in Banking

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Paper

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Preface

This paper has been written as part of my study Business Mathematics and Informatics (BMI) at the *vrije* Universiteit in amsterdam. The BMI-paper is one of the final compulsory subjects. The objective is to investigate the available literature in reference to a topic related to at least two out of the three fields integrated in the study.

During my internship with Rabobank International I became interested in the risk management process within banks. For the purpose of extending my knowledge of the available tools and techniques for measuring risks, I decided to dedicate my BMI paper to this field. The subject for this paper was set in consultation with dr. ir. G. Jongbloed of the Stochastics department of the faculty Exact Sciences.

Though several people have supported me during the realisation of this paper, I would especially like to express my gratitude to my supervisor, dr. ir. G. Jongbloed. Despite his busy schedule he made time to guide me. I am thankful for his advice, comments, and motivational speech when I needed it.

Purmerend, January 2005

Nazia Habiboellah.

Management Summary

Risk management within banking is intended to guard against risks of loss due to a fall in prices of the financial assets held or issued by the bank. Risk managers are primarily concerned with quantifying the effect of events that will have significant impact. Since these extreme events are very rare, not a lot of data is available about their occurrence and effect. Extreme Value Theory (EVT) is a specialised branch of statistics that attempts to make the best possible use of what little information is available about extreme events.

Though EVT has been around for some time now, the potential for the financial industry has only been recognised recently. In this paper an overview will be given of the application of extreme value theory as a method for modelling and measuring the extreme risks that are involved in financial mediation.

Two principal approaches can be distinguished within EVT, being the *block maxima* approach and the *peak-over-threshold* approach. The main difference between both approaches is the manner in which the extremes are identified. The fundamental theorem underlying the first approach states that the distribution of period maxima converges to the generalised extreme value distribution (Tippett - Fisher theorem). The second approach is based on the Balkema and de Haan - Pickands theorem, which states that the distribution of the peak over thresholds converges to the generalised Pareto distribution.

In the banking industry EVT is mainly used for assessing the Value at Risk of the investment portfolios, i.e. measuring the market risk exposure. Moreover, EVT can be applied in determining the adequate capital buffer to cover possible losses due to credit risk and operational risk, as prescribed by the Basle Capital Accord (BIS II).

List of Acronyms

ALM	Asset Liability Management
BIS	Bank of International Settlement
card(A)	Cardinal Number of the set A
EC	Economic Capital
EL	Expected Loss
ES	Expected Shortfall
EVT	Extreme Value Theory
FX	Foreign Exchange
GEV	Generalised Extreme Value Distribution
GPD	Generalised Pareto Distribution
iid	independent and identically distributed
LBO	Leveraged Buy Out
MDA	Maximum Domain of Attraction
MSE	Mean Square Error
POT	Peak Over Threshold
P&L	Profit and Loss
VaR	Value at Risk

Table of Contents

“If things go wrong, how wrong can they go?”

Introduction

> Banks are 'risk machines': they take risks, they transform them, and they embed them in banking products and services.
>
> [Besis, 2002]

Banks function as intermediaries in financial markets. On the one hand they attract capital from markets by issuing stock for example. On the other hand they invest capital back into the market by granting loans to individuals or corporations. This financial mediation involves various risks. For instance, when a debtor is not able to settle his debt at maturity of the loan, the bank will incur a loss. The possibility of this loss is referred to as credit risk. Other major risks in banking are market risk, operational risk, liquidity risk, and interest rate risk.

Risk managers are primarily concerned with the risk of rare events that could lead to catastrophic losses, like a market crash, or the default of a major international bank. Though the probabilities of these extreme events are very low, they could have a major impact on the economy. So, extreme events cannot be ignored for risk management purposes. The typical question that needs answering is: "When things go wrong, how wrong could they go?"

Over the last years a lot of research has been done on the utilisation of Extreme Value Theory (EVT) in banking. EVT is a specialised branch of statistics that attempts to make the best possible use of what little information is available about extreme events. Though EVT has been around for some time, the potential for the financial industry has only been recognised recently. In this paper an overview will be given of the application of extreme value theory as a method for modelling and measuring the extreme risks that are involved in financial mediation.

The structure of this paper is depicted in the overview below. After the introduction, risks and bank regulations are covered in chapter one. Then, a brief outline is given of the risk measures applied within the banking industry (chapter two). Thereupon, the fundamental theorems in EVT (chapter three), and the statistical techniques needed for applying EVT (chapter four) are discussed. Finally, the applications of EVT in the banking industry are treated in chapter five.

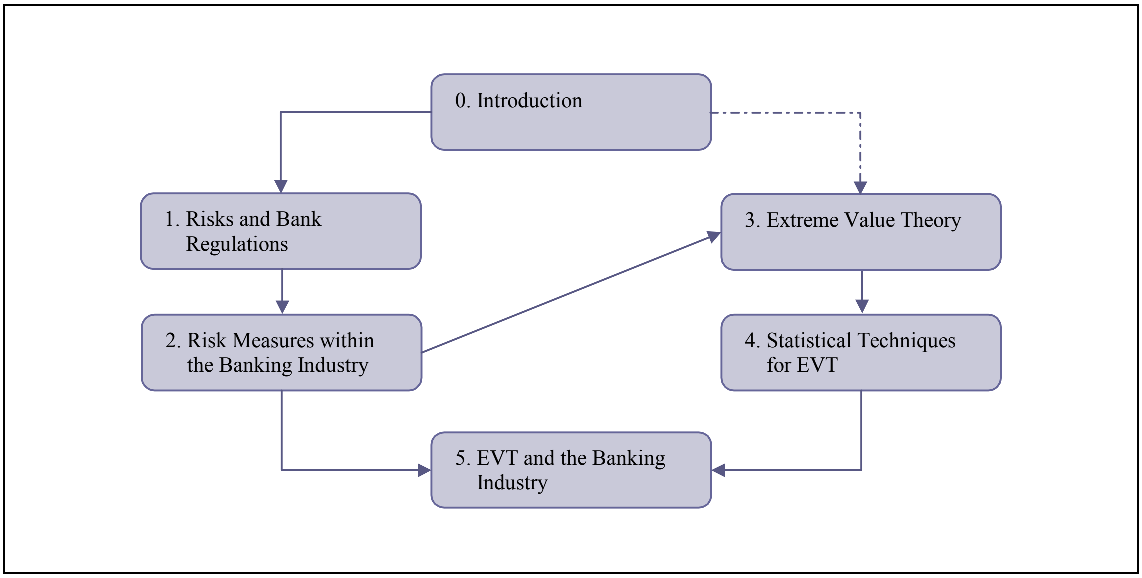


Figure 0.5: Paper Structure

Chapter 1. Risks and Bank Regulations

1.1 Introduction

This paper focuses on techniques for measuring extreme risks in the banking industry. These risks are important to manage carefully, for they could lead to significant losses. However, what are the extreme risks banks are exposed to, and how do risk managers in the banking industry manage these risk exposures? These questions will be answered in this chapter.

This chapter serves as general introduction to risk management in the banking industry. First paragraph 1.2 starts with a generic description of the term risk. Subsequently, the main risks in banking are outlined in paragraph 1.3, followed by a discussion on bank regulations in paragraph 1.4. In the latter, two types of capital buffers are distinguished, being regulatory capital and economic capital. Thereupon, the five phases of the risk management process are handled in paragraph 1.5.

1.2 What is Risk?

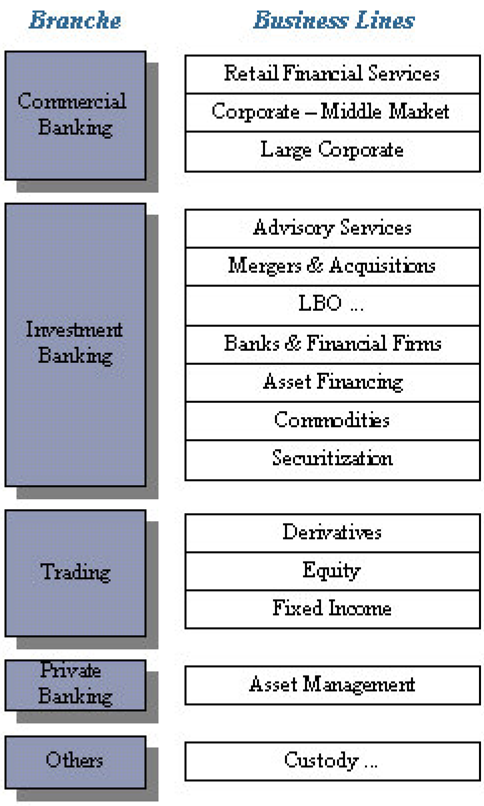
From a banking point of view, risk is the financial uncertainty that the actual return on an investment will be different from the expected return [http://www.datek.com/]. While in many risky situations the possible outcome can be classified either as a loss or a gain, generally only the 'downside' possibility of loss is considered to be risky, and not the upside 'potential' for gain.

In other words, risk designates any uncertainty that might trigger losses [Besis, 2002]. However, uncertainty is hardly visible in contrast to revenues or costs. Consequently, risks remain intangible until they have materialised into loss. This makes the quantification of risks more difficult.

1.3 Risks in Banking

The banking industry has a wide array of business lines, as can be seen in Figure 1.1. Each activity involves other types of risk. For instance, by lending money the bank is primarily exposed to credit risk, which is the risk that the borrower will default on his payment obligations. With trading on the other hand, market risk is the major risk involved, i.e. the risk on adverse market changes. An overview of the main bank risks and their definitions is given in Table 1.1.

Before proceeding, it has to be noted that not all risks mentioned in Table 1.1 can be assessed separately, for most risks interact. Market risk for example, is influenced by changes in interest rates and foreign exchange (FX) rates. So, it can be said that interest rate risk and FX risk are related to market risk. Another interesting detail is that credit risk is considered to be the most important risk in banking, for the losses suffered from this type of risk are generally much greater than the losses suffered from any of the other risk types.



Source: [Besis, 2002]

Figure 1.1: Overview of Banking Activities

Risk Type	Description
Credit risk	The risk that a company or individual will be unable to pay the contractual interest or principal on its debt obligations.
Foreign Exchange Risk	Foreign exchange risk applies to all financial instruments that are in a currency other than the domestic currency. When investing in foreign countries you must consider the fact that currency exchange rates can change the price of the asset as well.
Interest Rate Risk	The risk of (market) value changes that can be lost due to unexpected rate changes compared to the expected future value.
Liquidity Risk	The risk stemming from the lack of marketability of an investment that cannot be bought or sold quickly enough to prevent or minimise a loss.
Market Risk	The risk of adverse movements in market factors (such as asset prices, foreign exchange rates, interest rates) that cause volatility in profit and losses.
Operational Risk	The risk of loss resulting from inadequate or failed internal processes, people and systems or from external events.

Table 1.1: The Main Bank Risks Defined

1.4 Bank Regulations

The banking industry is a heavily regulated sector with respect to capital requirements. The objective of the capital regulation has always been to improve the safety of the banking industry, i.e. reducing the number of bank failures. The regulations developed by the Bank of International Settlement[1] (BIS) lay down principles of capital adequacy and risk-based capital. In other words, the regulations state that a banks' capital should be in line with its risk exposure.

In banking a distinction can be made between two types of capital, being regulatory capital and economic capital. Regulatory capital is determined by *general rules* prescribed to all banks by the Basle committee. In the new capital accord (BIS II)

[1] The Bank of International Settlements, located in Basle, Switzerland, was founded in 1930 and is an important forum for banking supervisors and central banks of the major industrialised nations to discuss and co-ordinate risk policies.

minimal capital requirements are stipulated for market risk, operational risk and credit risk (Pillar I). Economic capital is determined by the banks' *internal rules* for solvency and considers the 'actual' risk exposure of the bank. The internal rules are generally stricter and include more risk types.

The regulatory framework has a significant impact on the risk management process in the banking industry. The regulations stimulate the development and enhancement of the internal risk models and processes of banks. Moreover, banks are encouraged to develop better methodologies for measuring their own risk exposures in the economic capital framework. Thus, the quantitative assessment of risks is an important part of the risk management process.

1.5 The Risk Management Process

The risk management process provides a framework for identifying risks and deciding what to do about them. The process consists of five phases [Bodie, 2000]:

PHASE I: Risk Identification

In the first phase the most important risk exposures are determined for the unit under analysis. This can for instance be a company, or a stock or loan portfolio.

PHASE II: Risk Assessment

In the second phase, the costs associated with the risks that have been identified in the first phase are quantified. The remainder of this paper will mainly focus on this phase.

PHASE III: Selection of Risk Management Techniques

After the identification and quantification of the risk exposure, steps can be taken to reduce the risk. There are four basic techniques for reducing risk, being:

- *Risk Avoidance*
  Risk avoidance means that a conscious choice is made not to be exposed to a particular risk.
- *Loss Prevention*
  With loss prevention specific actions are taken to reduce the likelihood or the severity of losses.
- *Risk Retention*
  Risk retention means that the risks are absorbed by covering the losses with own resources.

- *Risk Transfer*
  This means that risks are transferred to a third party.

PHASE VI: Implementation

In this phase the selected techniques are implemented, with the restriction that the implementation costs have to be minimised.

PHASE V: Review

As time passes and circumstances change it will be necessary to periodically review and revise the decisions made. This is considered to be the final phase of the risk management process.

Chapter 2. Risk Measures within the Banking Industry

2.1 Introduction

Generally speaking, risk designates any uncertainty that might trigger losses. So, risks remain intangible and invisible until they materialise into loss. Therefore, risks are not always easy to identify and measure. Over the years, various risk measures have been developed that aim at capturing the variation of a given target variable due to uncertainty. Measures regarding market risk, for instance, record changes in the market value of assets, while credit risk measures consider the changes in losses caused by defaults.

This chapter focuses on the quantitative risk measures that are applied in banking. First, a general classification of the types of risk measures is handled in paragraph 2.2. Three categories are distinguished, being sensitivity indicators, volatility measures and downside measures of risk. Thereupon, paragraph 2.3 elaborates on a special group of downside risk measures, which are frequently used in banking, namely quantile based risk measures. The three risk measures that will be treated are Value-at-Risk, Expected Shortfall and Return Level.

2.2 Types of Risk Measures

According to Besis (2002), quantitative indicators of risk can be categorised into three groups, being sensitivity indicators, volatility measures and downside measures of risk. In this paragraph the characteristics of the measures in each of these groups will be discussed.

Sensitivity Indicators

A sensitivity indicator captures the deviation of a target variable due to the adverse change of a single parameter (the explanatory variable) by one unit. This type of indicator can link any target variable to the underlying sources of uncertainty that influence the target variable. This makes sensitivity indicators very convenient for risk measurement.

Sensitivities are widely used for measuring market risk and Asset-Liability Management (ALM). Market risk models, for instance use the ‘Greek letters’ to relate the market parameters, like the interest rates, to the target variable. ALM models however, use gaps. A gap is the sensitivity of interest income of the banking portfolio to shifts of interest rates.

Sensitivity indicators have several disadvantages. For one, they refer to a given adverse change of the risk drivers, such as a one percent shift of interest rates, without considering that some parameters are quite unstable while others are not. Thus, sensitivities do not take the deviation of parameters into account. Moreover, sensitivity indicators depend on the current conditions, like the value of the market parameters and the assets, making them proxies of actual changes.

Volatility Indicators

A volatility indicator measures the dispersion around the mean of any random parameter or target variable, taking both upside and downside deviations into account. Thus, unlike sensitivities, volatility indicators do consider the varying instability of uncertain parameters.

The volatility of a variable is given by the standard deviation of the values of the target variable. Given any set of data the (historical) volatility can be calculated. To this end no assumptions have to be made about the distribution of the target variable. Thus, to be able to calculation the volatility only a time series is required.

An important issue in defining the time series is the size of the sample. For short observation periods, or when just a few observations are available, the sampling error will be greater than for long observation periods with a lot of observations. This has to be kept in mind, during the computation.

Downside Measures of Risk

The purpose of downside risk measures is to capture losses, in which gains are ignored. So, volatility and downside risk are related, but not identical. In contrast to volatility, only the adverse changes are considered with downside risk. This leads to the following relation: when there is downside risk, there is also volatility. But when there is volatility, it does not naturally mean that there is a downside risk as is illustrated in Example 2.1.

Example 2.1

An option buyer is exposed to volatility without being exposed to downside risk, for the buyer only has an uncertain gain, and no risk on loss.

Suppose the buyer has a call option with a strike price of 100 dollars. At maturity of the option there are two possible scenarios:

1. The price of the underlying stock is higher than the strike price. In this case the buyer will exercise his option, and will gain the difference between the stock price and the strike price. This difference however, is currently uncertain and therefore leads to volatility.
2. The stock price is lower than 100 dollars. Under these circumstances the buyer will not exercise his option, and no loss is incurred. Thus, there is no downside risk.

Downside risk consists of two components, being the potential losses and the probability of occurrence. Though worst-case scenarios can be used to quantify the extreme losses, the chances of observing the scenarios are subjective. The relevancy or likelihood of a scenario changes with the individual's perception of the environmental uncertainty. For this reason, downside risk measures demand the prior modelling of the probability distributions of potential losses.

2.3 Quantile Based Risk Measures

Questions concerning risk management in banking often involve the estimation of extreme quantiles (refer to Example 2.2). With quantile estimation an adequate threshold can be determined that a target variable will exceed with a given (low) probability. Value-at-Risk (VaR) is the most frequently used quantile based risk measure in banking, and will be discussed in this paragraph. Furthermore, two other less popular measures will be handled, being the expected shortfall (ES) and the return level [Gilli, 2003].

Example 2.2

All banks are obliged to maintain a capital buffer to be able to cover possible future losses. The ability of a bank to cover the future losses is expressed in terms of a credit rating.

The Rabobank has a triple A rating, this is the highest existing credit rating. To be able to maintain the triple A status, the Rabobank has to reserve a capital buffer that reduces the probability of default to once in 10.000 years. This corresponds to determining the loss threshold that will be exceeded with a probability of 0.01 percent, i.e. the 99.99 percent quantile.

Value at Risk

Value at Risk (VaR) is defined as the maximum expected loss (measured in monetary units) of an asset value (or a portfolio) over a given time period and at a given level of confidence (or with a given level of probability), under normal market conditions [Coronado, 2000]. In other words, VaR indicates how much can be lost as a maximum, with a probability of 100p percent, during a period of time q. Example 2.2 for instance, refers to the VaR with p equal to 0.99 and q equal to one year.

Now, consider a random variable X that models losses or negative returns on a certain financial instrument over a certain time horizon. Given that X has a continuous distribution function F, the Value at Risk can be defined as the p-th quantile of the distribution F, with $0.95 \leq p \leq 1$. This can be written as

$$\text{VaR}_p = F^{-1}(p),$$

where F^{-1} is the inverse of the distribution function F, which is also known as the quantile function. The quantile function is formally defined as

$$Q(p) = F^{-1}(p) := \inf\{x : F(x) \geq p\}.$$

Despite of the general acceptance of VaR as a quantitative risk measure in banking, there has been some criticism on the reliability of the measure. According to Artzner et al. VaR is not a coherent risk measure[1]. In Artzner (1998) they have shown that the sub-

[1] For a discussion of the properties of coherent risk measures refer to Acerbi (2001) or Artzner(1997, 1998).

additivity[2] property does not always hold. This means that there are situations where the sum of the VaR of two separate portfolios can be less than the VaR of the two portfolios combined. In reality this cannot occur, for the total loss in a mixed bond portfolio can never be more than the sum of the losses on each bond in the portfolio[3]. Another point of criticism is that VaR does not give any information about the expected size of the loss, when the VaR is exceeded. Therefore, some practitioners prefer the use of the expected shortfall as an additional measure.

Expected Shortfall

Expected Shortfall (ES) is defined as the expected size of a loss that exceeds VaR, and is also referred to as the *tail conditional expectation*. Mathematically this risk measure can be defined as

$$\mathrm{ES_p} = \mathrm{E}(X \mid X > \mathrm{VaR_p}).$$

Even though the expected shortfall is currently not used that often, practitioners have realised its usefulness for financial risk management [McNeil, 1999].

Return Level

The return level (R_n^k) is a similar risk measure as Value at Risk, however, it is less frequently used in the banking industry. R_n^k designates the level that is expected to be exceeded, on average, only once in a sequence of k periods of length n. In Example 2.2 the size of the capital buffer could also be determined with the return level. In this case k is equal to 10.000, and n is one year.

Now, if H is the distribution of the maxima observed over successive non-overlapping periods of equal length, then R_n^k is the quantile

$$R_n^k = H^{-1}(p)$$

of the distribution function H. Since the event of interest only occurs once out of k periods, p is equal to $1/k$.

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2 A function ρ is said to be sub-additive when $\rho(X+Y) \leq \rho(X) + \rho(Y)$, where X, Y and $X + Y$ are elements of a set of real-valued random variables V. This property expresses the fact that a portfolio made of sub–portfolios will risk an amount that is at most the sum of the separate amounts at risk in its sub–portfolios.

3 However, due to diversification the loss in a mixed portfolio can be less than the sum of the losses of the bonds in the mixed portfolio.

Chapter 3. Extreme Value Theory

3.1 Introduction

Risk managers are mainly concerned with rare events that lead to significant losses. However, the occurrence of this type of events is very difficult to predict a long time ahead because there are not a lot of records of these events. EVT is a specialised branch of statistics that attempts to make the best possible use of the scarce information about extreme events, and provides a foundation for modelling them.

This chapter gives an introduction to the fundamental theorem in extreme value theory. Paragraph 3.2 discusses the general purpose of EVT, and distinguishes two principal approaches for modelling extreme events. Subsequently, the theorem underlying the first approach in extreme event modelling is discussed in paragraph 3.3, followed by a discussion of the second theorem in EVT in paragraph 3.4. And at last the challenges of applying EVT are handled in paragraph 3.5.

3.2 Extreme Value Theory

The fundamental theorem in extreme value theory is a cousin of the better-known central limit theorem. It defines what the distribution function of extreme events should look like in the limit, as the sample size increases. Consequently, EVT can be a helpful tool for finding the best possible estimate of the tail area of a distribution. Even when little historical data is available, EVT provides useful guidelines on selecting the type of distribution, to ensure that extreme risks are handled conservatively.

In extreme value theory, a distinction can be made between two approaches, namely the block maxima approach and the peak-over-threshold approach. The main difference between both approaches is the manner in which data is generated, i.e. how the extremes are identified. In Example 3.1 the data generation for both methods is illustrated by means of an example.

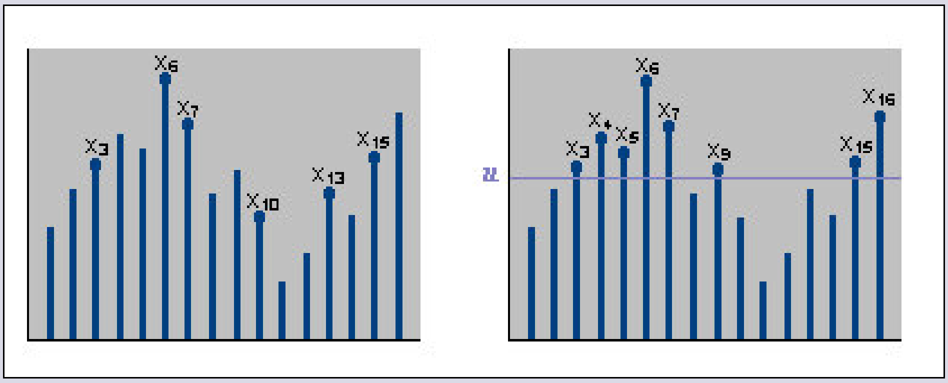
The block maxima approach is the oldest approach in EVT and considers the maxima (or minima) a variable takes in successive periods, like months or years. These selected observations are labelled extreme events, and are also referred to as block (or period) maxima. This method is the traditional method used to analyse time series with seasonality [Gilli, 2003]. In section 3.3 the fundamental theorem and the approach in modelling block maxima are outlined.
\
The second approach in EVT only considers observations that exceed a given threshold. This type of models is known as Peak-Over-Threshold (POT) models. These models use the data more efficiently than block maxima models, and are becoming the preferred method in recent applications. The underlying theorem of POT models and the approach in modelling are discussed in section 3.4.

Example 3.1

Consider a random variable representing daily losses and returns from an investment. The data is represented in the diagram below.

Utilizing the block maxima approach with six periods, each consisting of three months, the observations X_3, X_6, X_7, X_{10}, X_{13} and X_{15} are labelled as block maxima (see the left diagram below).

When the threshold method is applied with threshold *u*, the observations X_3, X_4, X_5, X_6, X_7, X_9, X_{15} and X_{16} are considered extreme events (refer to the right diagram below).



3.3 The Block Maxima Approach

Block maxima models consider the maxima of successive periods. This paragraph discusses the underlying theorem and gives a general distribution for block maxima. Furthermore, the concept of the maximum domain of attraction is illustrated by means of an example.

Limit Laws for Block Maxima

Suppose that $X = (X_1, \ldots, X_n)$ is a sequence of independent identically distributed observations with distribution function F, which does not necessarily have to be known. Then the sample maximum, M_n, with n the size of the sub sample (or block) is defined as $M_n = \max\{X_1, \ldots, X_n\}$. The limit law for block maxima is given by the following theorem:

Fisher - Tippett Theorem

Let (X_n) be a sequence of independent identically distributed random variables. If there exist norming constants $c_n > 0$, $d_n \in R$ and some non-degenerate distribution function H such that

$$\frac{M_n - d_n}{c_n} \xrightarrow{d} H,$$

then H belongs to one of the three standard extreme value distributions:

Fréchet: $$\Phi_\alpha(x) = \begin{cases} 0, & x \le 0 \\ e^{-x^{-\alpha}}, & x > 0 \end{cases} \qquad \alpha > 0,$$

Weibull: $$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^{\alpha}}, & x \le 0 \\ 1, & x > 0 \end{cases} \qquad \alpha > 0,$$

Gumbel: $$\Lambda(x) = e^{-e^{-x}}, \quad x \in R$$

The parameter α is the tail index, and reflects the degree of thickness of the tail of the distribution. In other words, the tail index measures the speed with which the tail approaches to zero, i.e. the heavier the tail, the slower the speed and thus the smaller the tail index.

The shape of the probability density functions for the standard Fréchet, Weibull and Gumbel distributions is given in Figure 3.1 below. For illustrative purposes the parameter α is set to 1.5 for both, the Fréchet and Weibull distribution.

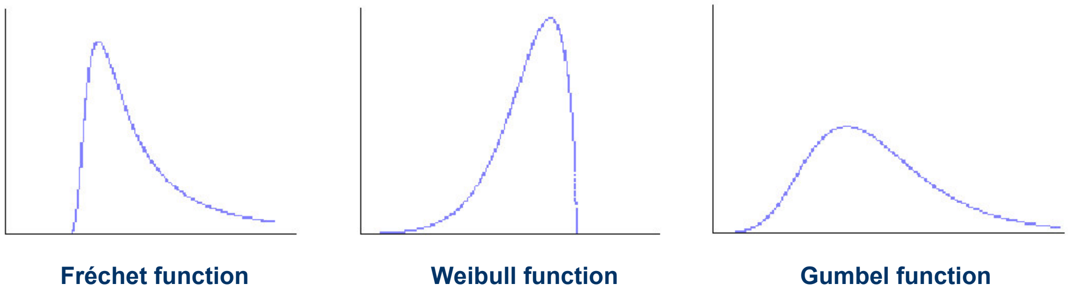


Figure 3.1: Extreme Value Distributions

The Generalised Extreme Value Distribution

As discussed in the previous section, there are only three standard extreme value distributions for modelling block maxima. However, the type of limiting distribution of the sample maxima is often not known in advance. Therefore, Jenkinson and von Mises suggested the following one-parameter representation of the three standard distributions

$$H_\xi(x) = \begin{cases} e^{-(1+\xi x)^{-1/\xi}}, & \text{if } \xi \neq 0, \\ e^{-e^{-x}}, & \text{if } \xi = 0, \end{cases}$$

where $1 + \xi x > 0$. This generalisation is known as the *generalised extreme value* (GEV) distribution, where the parameter $\xi = \alpha^{-1}$. This shape parameter determines the type of extreme value distribution, namely when:

- $\xi = \alpha^{-1} > 0$, the distribution takes the form of the Fréchet distribution Φ_α,
- $\xi = 0$, the distribution corresponds to the Gumbel distribution Λ,
- $\xi = -\alpha^{-1} < 0$, the distribution is known as the Weibull distribution Ψ_α.

This representation is nowadays widely accepted as the standard representation.

The Maximum Domain of Attraction

The extreme value distributions introduced in the previous sections represent the limit laws for maxima of independent identical distributed random variables. However, under what conditions do the normalised maxima M_n weakly converge to H? In other words, how do the constants $c_n > 0$ and $d_n \in \mathbb{R}$ have to be chosen such that

$$\frac{M_n - d_n}{c_n} \xrightarrow{d} H \; ?$$

In Example 3.2 an approach for finding the appropriate constants is illustrated for the case where the distribution function is known. It has to be noted that this approach can only be used when the expectation of the distribution exists. When the conditions are satisfied, it can be assumed that the normalised maxima M_n converge weakly to H.

The concept of maximum domain of attraction (MDA) is often used to characterise the convergence to an extreme value distribution. When the normalised maxima M_n converge weakly to H_ξ, the distribution function F is said to belong to the maximum domain of attraction of the extreme value distribution H_ξ. This is written as $F \in \text{MDA}(H_\xi)$.

Example 3.2

Consider a sample $\{X_1, X_2, \ldots, X_n\}$ of i.i.d. random variables from the standard exponential distribution $F(x) = 1 - e^{-x}$. Does the distribution function F belong to the MDA of the extreme value distribution H?

First the norming constants, c_n and d_n, have to be found. Then the limiting distribution of $(M_n - d_n)/c_n$ can be determined. To find the constant d_n that centres the distribution around zero, d_n has to be close to $E[M_n]$. An estimation of $E[M_n]$ can be obtained by the following approximation: $F(E[M_n]) \approx E[F(M_n)]$.

$F(X_1), F(X_2), \ldots, F(X_n)$ is a random sample from the uniform distribution over [0, 1]. Since $F(M_n)$ is the largest of the sample size, the probability density function of $F(M_n)$ is given by $f(M_n) = nx^{n-1}$. This gives $E[F(M_n)] = n/n+1$. Furthermore, $F(E[M_n]) = 1 - e^{-E[Mn]}$.

From this it can be deduced that $E[M_n] \approx \log(n+1) \approx \log n$ a reasonable choice for d_n is log n/n+1. The limiting distribution can now be determined as follows:

$$\begin{aligned}
\lim_{n\to\infty} P\left[\frac{M_n - d_n}{c_n} \le y\right] &= \lim_{n\to\infty} P\left[\frac{M_n - \log n}{c_n} \le y\right] \\
&= \lim_{n\to\infty} P\left[M_n \le c_n y + \log n\right] \\
&= \lim_{n\to\infty} \left[F(c_n y + \log n)\right]^n \\
&= \lim_{n\to\infty} \left[1 - e^{-c_n y - \log n}\right]^n \\
&= \lim_{n\to\infty} \left[1 - \tfrac{1}{n} e^{-c_n y}\right]^n \\
&= e^{-e^{-c_n y}}
\end{aligned}$$

The last equation is justified since $\lim_{n\to\infty}\left[1+\frac{a}{n}\right]^n = e^a$. Now, if the parameters are chosen as: d_n = log n and c_n = 1, then the limiting distribution is $e^{-e^{-y}}$, which is the Gumbel distribution. Therefore, it can be said that the exponential distribution belongs to the MDA of the Gumbel distribution.

3.4 The POT Approach

The second type of models in EVT only considers observations that exceed a given threshold. These models have largely been developed in the insurance business, where only losses above a certain threshold are accessible to the insurance company. In this paragraph the main theorem underlying this approach is discussed and a general distribution for the exceedances is given.

The Distribution of Exceedances

Suppose that $X = (X_1, \ldots, X_n)$ is a sequence of independent identically distributed observations with distribution function F, which does not necessarily have to be known. Then the threshold u has been exceeded when $X_i > u$. This excess over u, also referred to as peak over threshold, is defined by $y = X_i - u$. The distribution of the excess losses over the threshold u is given by

$$F_u(y) = P(X - u \leq y \mid X > u),$$

for $0 \leq y < x_F - u$ where $x_F \leq \infty$ is the right endpoint of the distribution function F.

The distribution F_u is called the conditional excess distribution function and represents the probability that the value of X exceeds the threshold by at most an amount of y given that X exceeds the threshold u. This conditional probability can be written as

$$F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)}.$$

Since the realisations of the random variable X mainly lie between 0 and u, the estimation of F in this interval is generally not that difficult. The estimation of F_u

however is more difficult as in general very little observations are available in this area. The peak over threshold approach offers a solution for this problem.

Limit Laws for Peak Over Thresholds

The Fisher-Tippett theorem, discussed in the previous paragraph, is the basis for the theorem for peak over thresholds. The theorem by Balkema and de Haan (1974) and Pickands (1975) shows that for a sufficiently high threshold u the distribution function of the excesses can be approximated by the generalised Pareto distribution $G_{\xi,\beta(u)}$.

Balkema and de Haan – Pickands Theorem

For every $\xi \in R$, $F \in MDA(H_\xi)$ if and only if

$$\lim_{u \uparrow x_F} \sup_{0<x<x_F-u} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0$$

for some positive function β..

Besides the magnitude of the losses, the POT approach also models the occurrence of the losses. Resulting from the properties of the GPD as discussed by Embrechts et al. (1997), it can be said that the number of excesses of the threshold follows a Poisson process. However, this property will not be discussed any further in this paper.

The Generalised Pareto Distribution

The generalised Pareto distribution (GPD) is the limiting distribution for the peak over threshold approach and is defined as

$$G_{\xi,\sigma,\upsilon}(x) = \begin{cases} 1-\left(1+\xi\dfrac{x-\upsilon}{\beta}\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1-e^{-(x-\upsilon)/\beta} & \text{if } \xi = 0 \end{cases}$$

with

$$x \in \begin{cases} [\upsilon, \infty], & \text{if } \xi \geq 0 \\ [\upsilon, \upsilon - \beta/\xi], & \text{if } \xi < 0 \end{cases}$$

where $\xi = 1/\alpha$ is the shape parameter, α is the tail index, β is the scale parameter, and ν is the location parameter. When $\nu = 0$ and $\beta = 1$, the representation is known as the standard GPD.

The generalised Pareto distribution embeds three other distributions. The type of distribution is determined by the shape parameter (just like with the GEV distribution). When:

- $\xi = \alpha^{-1} > 0$, the distribution takes the form of the ordinary Pareto distribution,
- $\xi = 0$, the distribution corresponds to the exponential distribution,
- $\xi = -\alpha^{-1} < 0$, the distribution is known as a Pareto II type distribution.

In Figure 3.2 the shape of the probability density functions of above-mentioned distributions are illustrated. The standard distributions are depicted, which means that the location parameter ν is set to 0, and the scale parameter β is equal to 1. The corresponding distribution functions are represented in Figure 3.3.

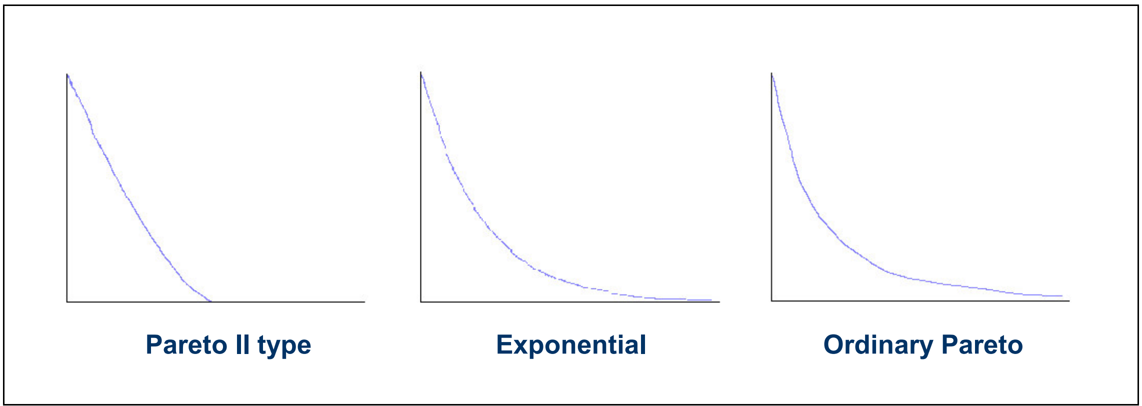


Figure 3.2: Densities for the Generalised Pareto Distributions

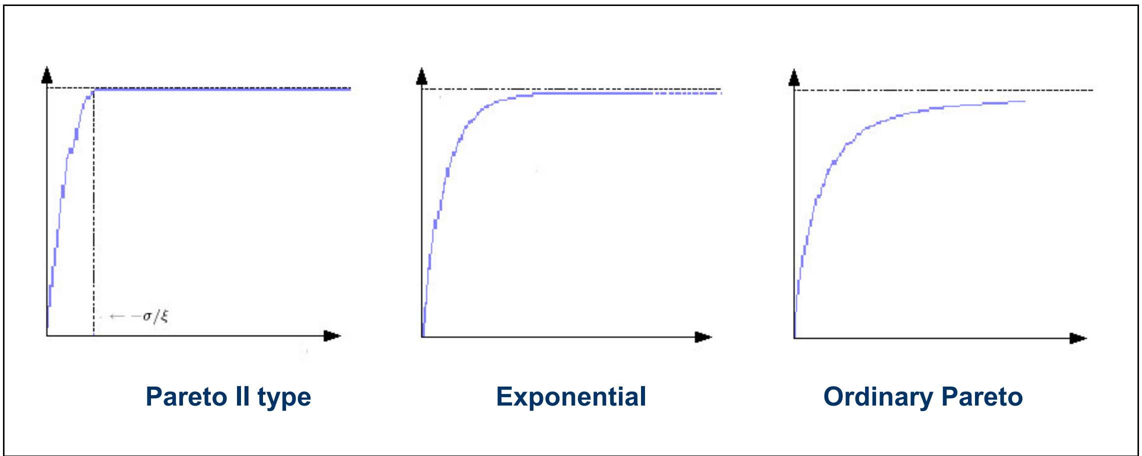


Figure 3.3: Generalised Pareto Distributions

3.5 Challenges of Applying EVT

In risk management, EVT is considered to be a useful tool for measuring extreme risks. However, the application of EVT involves a number of challenges. Since the availability of extreme data is limited, determining whether the series is “fat-tailed”, choosing the threshold or beginning of the tail, and choosing the methods of estimating the parameters is more difficult. Therefore, it is very important to analyse the data carefully, before proceeding.

In order to apply EVT sufficient data should be available for estimating the parameters of the limiting distributions. When the POT approach is adopted, the following assessment has to be made in determining an adequate threshold u: u has to be large enough to satisfy the conditions that allow the convergence to the limiting distribution, while plenty of observations remain for estimating the parameters. This can also be said for the block maxima method with regard to the size of the blocks.

Furthermore, in traditional EVT it is assumed that the extreme observations are independent and identically distributed. This assumption does not always hold, for in reality extreme events often tend to occur in clusters. This is caused by the dependence in the data. With the block maxima approach increasing the size of the blocks can reduce this dependence. However, as a result some extreme observations can be lost for they occurred in the same block. This assessment makes the choice of the block size more difficult.

There are some additions to the theory of estimating the parameters for dependent observations, however, these are not considered to be in the scope of this paper. For more information on modelling dependent observations refer to Embrechts et al. (1997).

Chapter 4. Statistical Techniques for EVT

4.1 Introduction

The implementation of extreme value theory faces many challenges. For one, data of extreme events is very scarce, which makes the process of estimating the appropriate distribution function more difficult. The assessment whether or not the series is fat-tailed, and the determination of the threshold or beginning of the tail are examples of important issues that need to be tackled. Moreover, choosing the methods of estimating the parameters is not that easy either.

In this chapter the necessary statistical techniques for applying EVT are discussed. In paragraph 4.2 two useful graphical techniques are discussed for exploring the behaviour of the tail of the empirical distribution, namely the QQ-plot and the mean excess plot. Paragraph 4.3 explains how the graph of mean excess and the hill graph can be used to determine the threshold of the GPD. Thereupon various techniques are given in paragraph 4.4 to estimate the parameters for both the GEV distribution and the GPD. And finally, paragraph 4.5 illustrated how the confidence intervals can be assessed.

4.2 Graphical Data Exploration Tools

The first and foremost step in modelling is the (graphical) exploration of the data [Bensalah, 2000]. In this phase the modeller has to become familiar with the features of the data that are relevant for the question at hand. There are several graphical techniques for the exploration of data. The most common techniques include histograms and box plots. This paragraph however, concentrates on techniques that provide as much information as possible about the tail of the distribution. The graphs that are discussed are QQ plots and mean excess plots.

Quantile Quantile Plots

Quantile Quantile Plots, better known as QQ plots, depict the relation between the quantiles of an empirical distribution and a standard distribution. The graph displays a linear line when the quantiles of a class of distributions are related to the corresponding quantiles of the empirical distribution. This tool is easy to use, since the linearity in the graph can be easily checked by eye. When necessary the relation can further be quantified by means of the correlation coefficient[1].

Considering a sequence of iid random variables $X_1, \ldots, X_n$ with empirical distribution F_n, and $X_{n,n} < \ldots < X_{1,n}$ the order statistics, it can be said that $F_n(X_{k,n}) = (n-k+1)/n$. Given that F denotes the estimated parametric distribution of the data, the graph of quantiles is defined by the set of the points:

$$\left\{ X_{k,n}, F^{-1}\left(\frac{n-k+1}{n}\right), k = 1, \ldots, n \right\}.$$

The more linear the QQ plots, the more appropriate the model in terms of goodness of fit. In Figure 6.1 two QQ plots are depicted. The left graph illustrates a good fit, in this case the empirical distribution belongs to the normal family. In the right graph a fat tailed distribution is illustrated. This can be derived from the curves to the top at the right end and/or to the bottom at the left end.

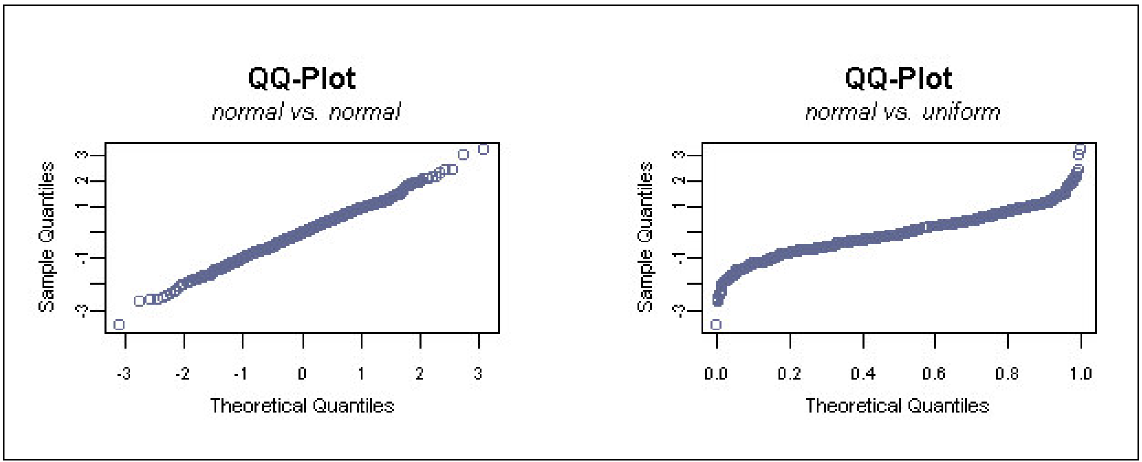


Figure 4.1: Quantile Quantile Plots

[1] The correlation coefficient r represents the relation between two variables, where $-1 \leq r \leq 1$. When $r = \pm 1$ the points lie perfectly on a straight line.

Thus, QQ plots are easy and useful tools for determining the distribution of the data. They can also be used to assess how well a model fits the tail of an empirical distribution.

Mean Excess Plots

The mean excess is the expected size of the excess over a given threshold u, given that u is exceeded. The mean excess function *e(.)* for a random variable X is defined as

$$e(u) = E(X - u \mid X > u), \quad \text{for } 0 \le x \le x_F \,,$$

where x_F is the upper bound of the distribution. The particular case *e(u)* is called the mean excess over the threshold u. In financial risk management *e(u)* is better known as the Expected Shortfall (as discussed in section 2.3). For an elaborate discussion of the properties of this function refer to Embrechts et al. (1997).

The behaviour of the tail can now be established based on the form of the distribution of mean excesses. Let $X_1, \ldots, X_n$ be iid with empirical distribution F_n. Then,

$$e(u) = \frac{1}{card\Delta_n(u)} \sum_{i \in \Delta_n(u)} (X_i - u), \quad u \ge 0$$

where $\Delta_n(u) = \{i, i = 1, \ldots, n, X_i > u\}$ and the *card* function counts to the number of points in the set $\Delta_n(u)$. So, the mean excess function is nothing else than the excesses over the threshold u divided by the number of data points that exceed the threshold u.

Now, the mean excess graph is formed by the following set of points:

$$\{X_{k,n}, e_n(X_{k,n}), k = 1, \ldots, n\}.$$

The mean excess graph will tend towards infinity for fat-tailed distributions, i.e. the graph will have a linear shape with a positive slope when the data follows the GPD with a positive shape parameter (ξ). On the other hand, when the data is exponentially distributed the graph will show a horizontal line, and for short-tailed data, the line will have a negative slope. In Figure 4.2 the mean excess graph is illustrated for a fat tailed (left illustration) and the exponential (right illustration) distribution. For an overview of the most important mean excess functions refer to Embrechts et al. (1997).

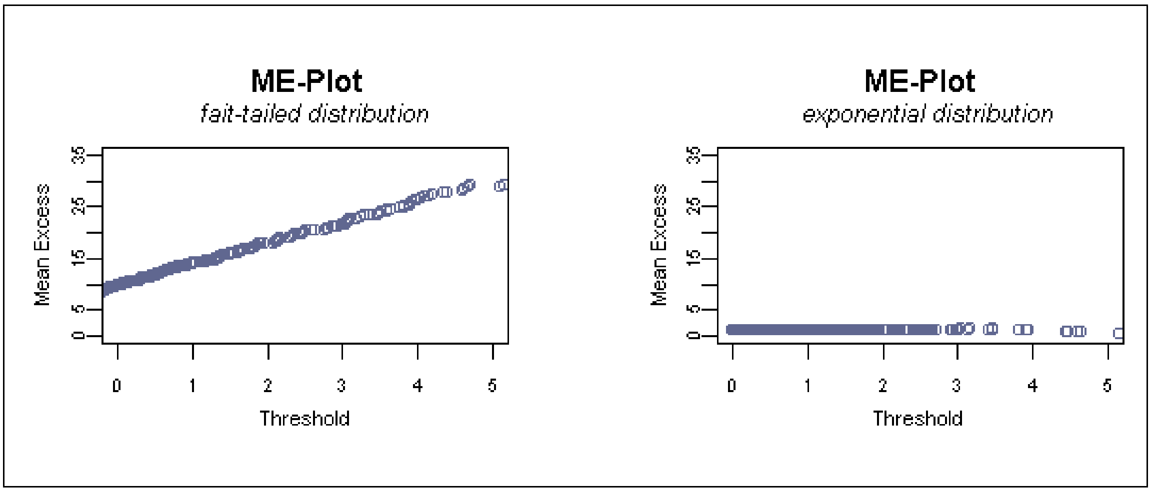


Figure 4.2: Mean Excess Plots

4.3 Determining the Threshold

An important step in modelling the POT is the determination of an appropriate threshold value u. The assessment has to be made whether u is large enough to satisfy the conditions that allow the convergence to the limiting distribution, while leaving plenty of observations for estimating the parameters. In this paragraph two graphical tools will be discussed to determine an appropriate threshold, being the graph of mean excess and the hill graph.

Graph of Mean Excess

The graph of mean excess has already been discussed in the previous section. Besides determining the type of distribution, this graph is also a helpful tool for choosing an appropriate threshold. The Balkema and de Haan – Pickands theorem states that for a high threshold, the excess over the threshold for a given series converges to a GPD. Since the mean excess graph of the GPD is linear, it is possible to choose the threshold by detecting an area with a linear shape on the graph.

Hill Graph

Another helpful tool for determining the threshold is the Hill plot. Given that $X_{n,n} < \ldots < X_{1,n}$ are the ordered statistics of iid random variables, the Hill estimator of the tail index α is defined by:

$$\hat{\alpha}^{(H)} = \hat{\alpha}_{k,n}^{(H)} = \left(\frac{1}{k} \sum_{j=1}^{k} \ln X_{j,n} - \ln X_{k,n} \right)^{-1},$$

where $k \to \infty$ is the number of upper order statistics and n is the sample size. A Hill plot is constructed such that the estimated α is plotted as a function of k upper order statistics (or the threshold). Thus, the Hill graph is defined by the set of points

$$\left\{ \left(k, \hat{\alpha}_{k,n}^{(H)} \right), \quad 1 \le k \le n-1 \right\}.$$

The threshold is selected from this graph for the stable areas of the tail index. However, this choice is not always clear. The ambiguity of the value of threshold leads to some difficulty. In determining the threshold the variance has to be considered against the bias. If a low threshold is chosen, the number of observations (exceedances) increases and the estimation becomes more precise. However, by choosing a low threshold also some observations from the centre of the distribution are included, what makes the estimation become biased.

In conclusion, the estimates of α based on a few of the largest observations are highly sensitive to the number of observations used (high volatility). But when too many elements are used the estimation becomes biased. Therefore, a combination of the aforementioned techniques should be considered for determining an appropriate threshold.

4.4 Parameter Estimation

The parameters of the GPD or the GEV distribution can be estimated in various ways. In this section three techniques will be outlined, being maximum likelihood estimation, the method of moments and the method of probability weighted moments. In practice, the first mentioned technique is used the most.

Maximum Likelihood Estimation

One way of estimating the parameters of the distribution is by following the maximum likelihood methodology. Suppose that $X = (X_1, \ldots, X_n)$ is a random variable with density p_θ. Then the likelihood function is given by

$$\mathrm{L}(\theta; X) := \prod_{i=1}^{n} p_\theta(X_i).$$

The log likelihood function is denoted by $l(\theta;\ X) := \ln L(\theta;X)$. The Maximum Likelihood Estimator (MLE) for θ is the value that maximises the log likelihood function [Oosterhof, 1999].

The big advantage of maximum likelihood procedures is that they can be generalised, with very little change in the basic methodology, to much more complicated models in which trends or other effects may be present [Embrechts, 1997].

The Method of Moments

This technique implies that the theoretical moments based on H_θ (model based) are equated with the corresponding empirical moments based on the data. This comes down to the following equation:

$$\int_{-\infty}^{+\infty} x dH_\theta(x) = \int_{-\infty}^{+\infty} x dF(x),$$

where F denotes the empirical distribution function of the data. However, this method is considered to be very unreliable, for the second- and higher-order moments cannot always be determined [Embrechts, 1997].

Method of Probability-Weighted Moments

This technique is considered to be more promising than the method of moments discussed above. It is simple to apply and performs well in simulations. The probability-weighted moment is defined as

$$\omega_r(\theta) = E(X\, H_\theta^r(X)), \quad r \in \mathrm{N}_0$$

where X has distribution function H_θ. Now, in order to estimate θ the empirical estimate of ω has to be equated with the theoretical analogue of ω. This can be written as:

$$\int_{-\infty}^{\infty} x H_\theta^r(x) dF(x) dx = \int_{-\infty}^{\infty} x H_\theta^r(x) dH(x) \quad r = 0, 1, 2$$

where F denotes the empirical distribution function.

Chapter 5. EVT and the Banking Industry

5.1 Introduction

Extreme value theory provides some ready-made approaches for modelling events that are both extreme, and extremely rare. A distinction can be made between two principal kinds of models, being block maxima models and peak-over threshold models. The former considers maxima of successive periods. And the latter only considers observations that exceed a given threshold.

Though EVT has been around for some time now, the potential of EVT for the financial industry has only been recognised recently. This chapter focuses on the application of EVT in the banking industry. In paragraph 5.2 the importance of EVT in managing financial risk is discussed, followed by several examples of the utilisation of EVT in banking in paragraph 5.3. Finally, the implementation of EVT is illustrated by means of an example in paragraph 5.4.

5.2 Why EVT?

Risk management in the banking industry is intended to guard banks against risks of loss due to a fall in prices of financial assets held or issued by the bank. One of the major challenges to risk managers is the implementation of risk management models that allow for extreme events, and permit the measurement of their far-reaching or even fatal consequences [McNeil, 1999].

Classical data analysis techniques are inadequate for this purpose, because they cut off extreme data, i.e. the extreme values are often labelled as outliers and sometimes even ignored. EVT however, offers two general distribution functions for modelling these extreme values. The general extreme value distribution models maxima, while the general Pareto distribution models the exceedances above a given threshold.

5.3 Applications in Banking

In banking extreme value theory is used for managing all three risks for which regulations are prescribed, being credit risk, market risk and operational risk. For market risk purposes EVT is used to determine the value at risk (VaR) for the losses incurred on the trading book due to adverse changes in the market. BIS II stipulates that banks should be able to cover losses on their trading portfolios over a ten-day horizon, 99 percent of the time. However, financial institutions generally compute a 5 percent VaR over a one-day holding period for their internal risk management purposes [Gilli, 2003].

In credit and operational risk management EVT is often utilised for determining the adequate level of risk capital, that serves as a buffer against irregular losses from respectively defaults, or operational problems. The economic capital buffer is maintained to be able to absorb the losses exceeding the expected loss (banks build provisions to cover the expected losses). The size of the EC buffer depends on the credit rating of the bank (refer to Example 2.2), and can be determined as: EC = VaR_p – EL, where p is the confidence level.

5.4 A Practical Example

In this section the Value at Risk, Return Level and Expected Shortfall will be determined for the S&P 500 index. This index is especially designed to measure the performance of the domestic economy through changes in the aggregate value of 500 stocks that represent all major industries. The analysis is performed with the help of the software package R[1] (refer to Appendix A for the used R code).

The dataset used for analysis consists of the closing values of the Standard & Poors 500 index from January 5 1960 to October 16 1987, resulting in 6985 observations. The time-plot of the S&P 500 index is given in Figure 5.1, and Figure 5.2 depicts the daily log returns. The log returns are defined as $r_t = \log(p_t/p_{t-1})$, where p_t denotes the value of the index at day t for $t = 2, \ldots, 6985$.

[1] R is a language and environment for statistical computing and graphics similar to S-plus. R can be downloaded free of costs from http://www.r-project.org/.

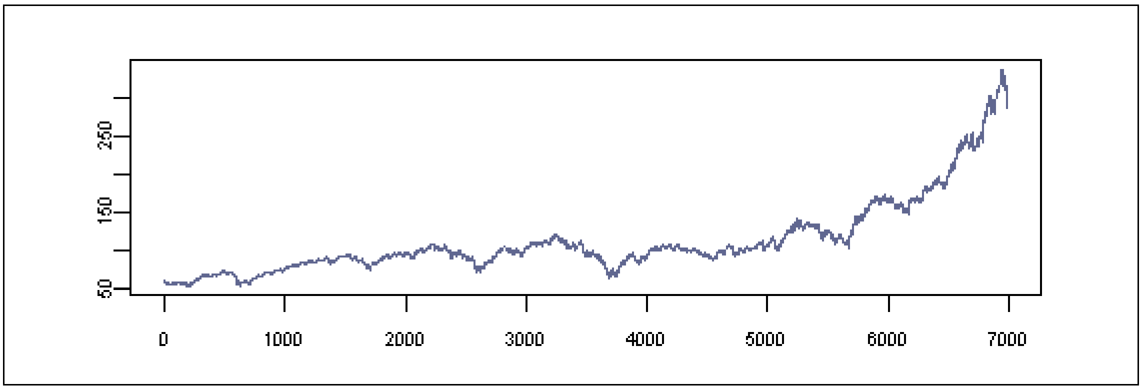


Figure 5.1: Daily Quotes of the S&P Index 500

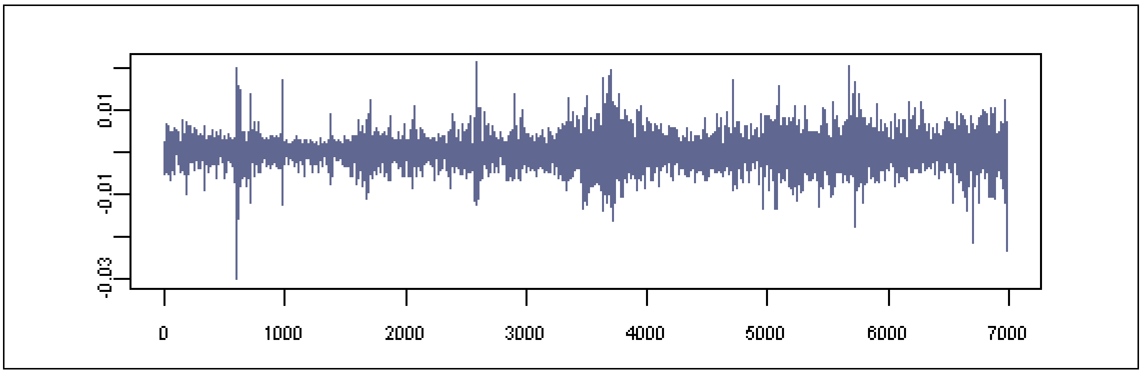


Figure 5.2: Daily log Returns of the S&P Index 500

Now the distribution of the loss data can be approached with the normal distribution. In Figure 5.3 the histogram of the data is given together with the fitted normal distribution with parameters $\hat{\mu}$ = -0.00009 and $\hat{\sigma}$ = 0.00350. The diagram on the right side zooms in on the right tail of the distribution, and illustrates that the distribution has a heavier tail than the normal distribution.

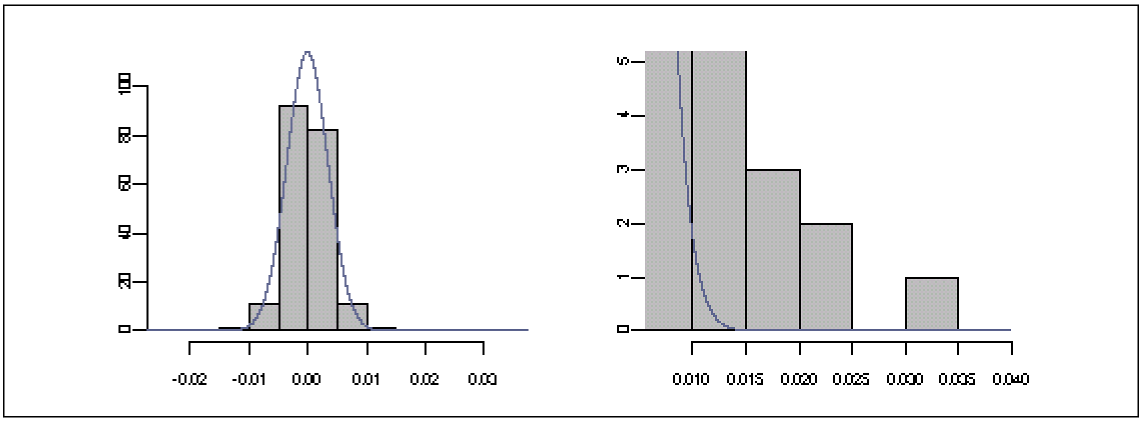


Figure 5.3: Approximation of Distribution with Normal Distribution

The same conclusion can be drawn from the QQ plot and the mean excess plot in Figure 5.4. The line in the QQ plot slightly deviates to the bottom in the left tail and to the top in the right tail, which is an indication of heavy tails. The Mean Excess plot also points to a heavy tailed distribution, for the line has a positive slope for thresholds greater than zero.

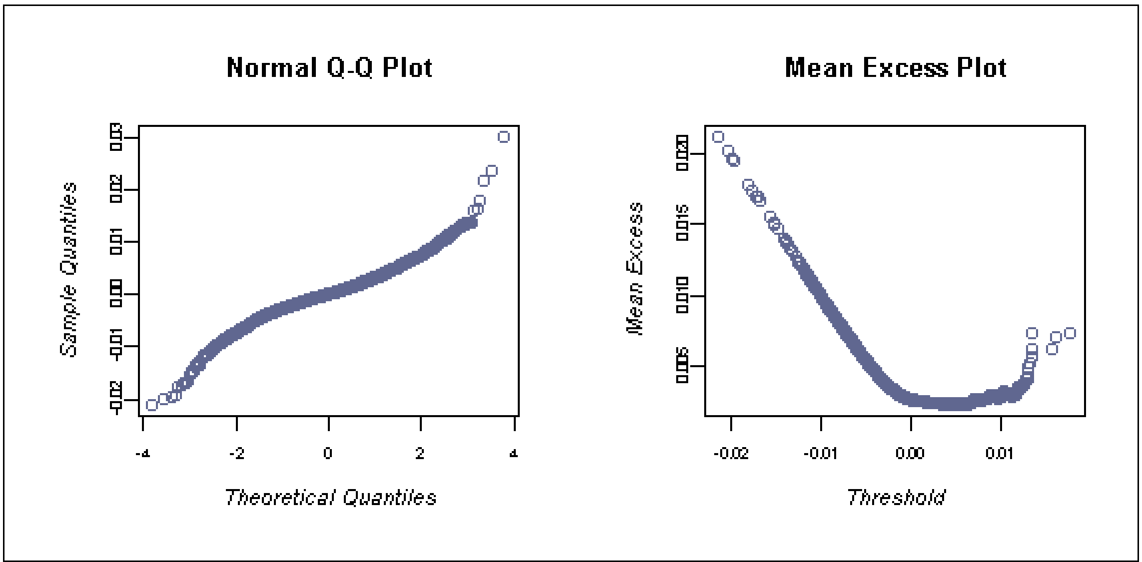


Figure 5.4: Exploring the Tails

Block Maxima Method

The block maxima method can be used to approximate the limiting distribution of the maxima. The first step is to determine the appropriate block size *N*, so that *N* is large enough to fulfil the conditions that allow the convergence to the GEV distribution, while sufficient observations remain to estimate the parameters for the limiting distribution.

In Table 5.1 the parameter estimates of the limiting extreme value distribution are given for six time periods with different lengths. The standard errors of the estimates are also given in the table (noted between brackets). The best approximation can now be determined by considering the variance of the estimations (i.e. the standard error) and by studying the distribution of the residuals. The QQ plots of the residuals (refer to Appendix B) show that the models based on all selection periods, except for the daily returns, result in an adequate approximation of the distribution.

When the estimated distributions are plotted with the histogram of the data, the weekly returns seem to give the best approximation of the tail area (refer to Figure 5.5). The other estimated distributions have thicker tails than the empirical distribution.

Selection period		Shape Parameter (ξ)	Location Parameter (μ)	Scale Parameter (σ)
Daily returns	N = 1	-0.1203	-0.0016	0.0039
(st. error)		(1.97E-06)	(1.97E-06)	(1.97E-06)
Weekly returns	N = 5	0.0518	0.0021	0.0021
(st. error)		(1.94E-02)	(5.67E-05)	(2.01E-06)
Monthly returns	N = 21	0.0965	0.0045	0.0022
(st. error)		(3.97E-02)	(1.16E-04)	(2.00E-06)
Quarterly returns	N = 63	0.2221	0.0061	0.0023
(st. error)		(8.90E-02)	(2.01E-04)	(2.01E-06)
Semester returns	N = 125	0.3422	0.0073	0.0024
(st. error)		(2.29E-01)	(3.68E-04)	(2.25E-06)
Yearly returns	N = 250	0.3451	0.0087	0.0030
(st. error)		(2.57E-01)	(5.16E-04)	(2.00E-06)

Table 5.1: Parameter Estimation for GEV Distribution with Increasing Block Size

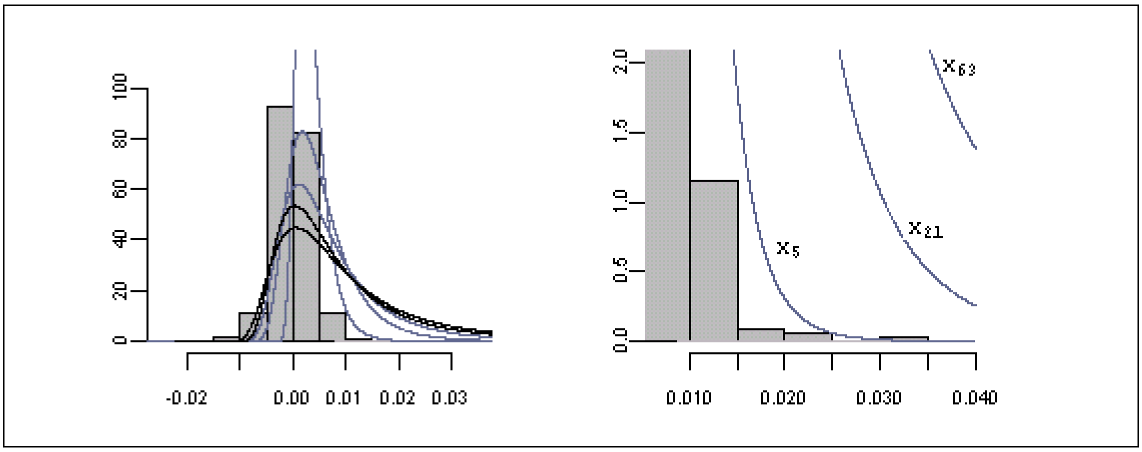


Figure 5.5: Limiting Distribution for Empirical Data

The limiting distribution of the weekly maxima approximates the empirical distribution best. Therefore, the tail of the empirical distribution can be approached with the Fréchet distribution ($\xi > 0$) with shape parameter 0.0518, scale parameter 0.0021 and location parameter 0.0021. Now the Return Level can be determined. In Table 5.2 the *k*-block Return Level with corresponding 95% confidence interval is given for various periods. For instance, once every month the incurred losses will exceed the 0.0048. And once every quarter the losses will be higher than 0.0078.

k-blocks	Return Level	95%confidence interval
4	0.0048	[0.0047, 0.0048]
13	0.0078	[0.0075, 0.0082]
26	0.0095	[0.0091, 0.0102]
52	0.0113	[0.0107, 0.0121]

Table 5.2: k-block Return Level with Length 5

Peak Over Threshold Approach

The Expected Shortfall and the Value at Risk can be estimated by means of the peak over threshold approach. The most important step is the determination of an adequate threshold. For this purpose the Mean Excess plot and the Hill graph can be used. Both graphs are depicted in Figure 5.6 below.

From the Mean Excess plot can be derived that the threshold lies between 0.005 and 0.010, for the graph shows a linear line with a positive slope in this interval. By exploring the Hill graph the boundaries of the interval can be further specified. Since the Hill graph is stable for alpha when 70 to 98 order statistics are used, the interval of the threshold can be narrowed down to 0.0082 and 0.0089. As an additional option the threshold of 0.0077 (124 order statistics) is also taken into account in the analysis.

The parameter estimates of the GPD for the three threshold options specified above are given in Table 5.3. The standard deviations of the estimates are given between brackets.

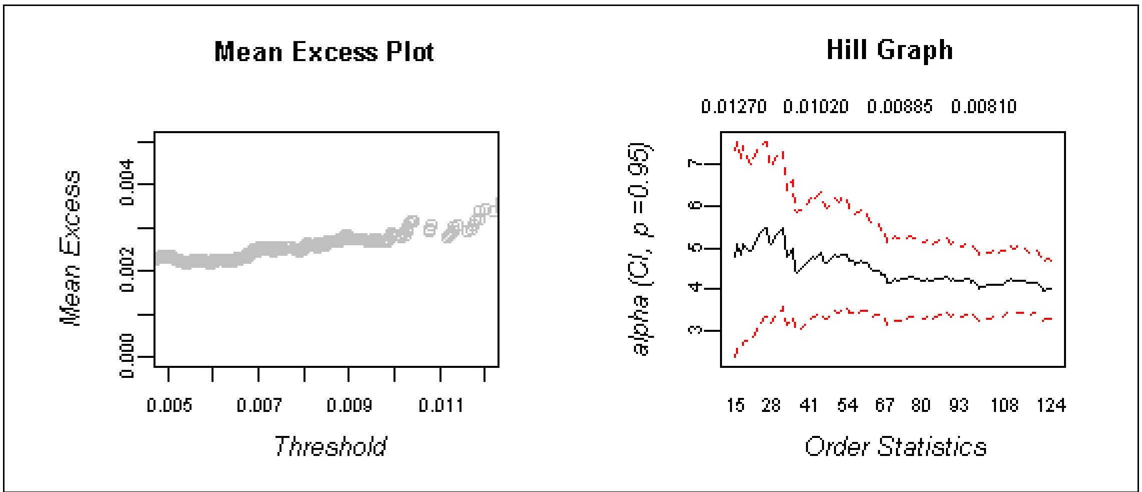


Figure 5.6: Determining the Threshold

Threshold (u)	Parameters		
	Shape (ξ)	Location (ν)	Scale (β)
0.0089	0.1572	0	0.0023
(st.dev.)	*(0.1083)*		*(0.0002)*
0.0082	0.1521	0	0.0022
(st.dev.)	*(0.0909)*		*(0.0001)*
0.0077	0.1416	0	0.0021
(st.dev.)	*(0.0793)*		*(9.27e-05)*

Table 5.3: Parameter Estimation for GPD for Various Thresholds

From Figure 5.7 can be derived that the model based on the first 124 order statistics approaches the distribution of the data the most accurate. Thus, the tail of the empirical distribution can be approximated by the ordinary Pareto distribution (since $\xi > 0$), with location parameter 0, and scale parameter 0.0021. In Table 5.4 the Value at Risk and Expected Shortfall are given with their corresponding confidence interval for several confidence levels.

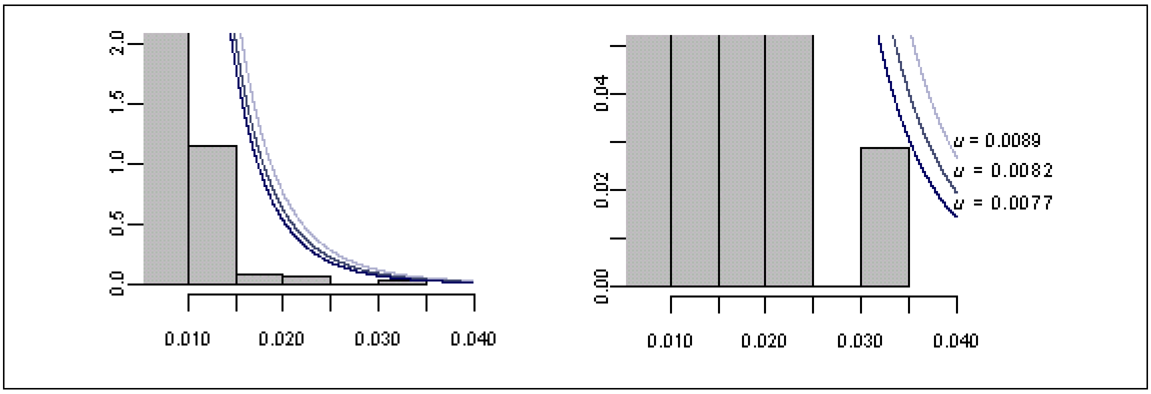


Figure 5.7: Limiting Distribution for Empirical Data

p	VaR_p	95% Confidence Interval	ES_p	95% Confidence Interval
0.925	0.0144	[0.0132, 0.0164]	0.0180	[0.0157, 0.0238]
0.95	0.0157	[0.0142, 0.0185]	0.0195	[0.0168, 0.0270]
0.975	0.0181	[0.0159, 0.0228]	0.0223	[0.0185, 0.0340]
0.99	0.0216	[0.0183, 0.0306]	0.0264	[0.0208, 0.0450]
0.999	0.0329	[0.0240, 0.0450]	0.0395	[0.0264, 0.0450]
0.9999	0.0484	[0.0295, 0.0450]	0.0576	[0.0319, 0.0450]

Table 5.4: Var and ES for Different Confidence Levels

Glossary

Bank of International Settlement	The Bank of International Settlements, located in Basle, Switzerland, was founded in 1930 and is an important forum for banking supervisors and central banks of the major industrialised nations to discuss and co-ordinate risk policies.
Banking Book	The banking book groups all commercial banking activities. It includes all lending and borrowing, usually both for traditional commercial activities, and overlaps with investment banking operations. Accounting rules for the banking book use accrual accounting of revenues and costs, and rely on book values for assets and liabilities.
Basle Committee	*See* Bank of International Settlements
BIS II	Also known as Basle II, is the new Basle Capital Accord. The new regulations will come into force in 2007. However, they are to be implemented by banks before the beginning of 2006.
Block Maxima Method	This approach in EVT considers the period maxima in modelling extreme events.
Commercial Banking	Commercial banking entails all traditional banking activities, being lending and collecting deposits from individuals and small businesses, and so-called relationship banking.

Credit Risk	The risk that a company or individual will be unable to pay the contractual interest or principal on its debt obligations.
Default	A debtor is said to be in default when he fails to meet a contractual obligation, such as the repayment of either principal or interest.
Diversification	Diversification is a risk-reduction technique. By investing in a range of different investments the overall portfolio risk can be reduced. That is because different types of investments tend to behave differently under the same market conditions. By holding a range of investments the chance that when some investments in the mix are declining, others may be rising can be increased.
Downside Risk	Downside risk refers to the possibility of losses in which the possibility on gains is ignored. Downside risk consists of two components, being the potential losses and the probability of occurrence.
Economic Capital	Economic Capital is the capital buffer that is reserved based on the internal capital requirements specified by the bank itself. The Economic Capital considers the 'actual' risk of the bank.
Expected Shortfall	The expected size of a loss that exceeds VaR (also referred to as the tail conditional expectation).
Foreign Exchange Risk	Foreign exchange risk applies to all financial instruments that are in a currency other than the domestic currency. When investing in foreign countries you must consider the fact that currency exchange rates can change the price of the asset as well.
Interest Rate Risk	The risk of (market) value changes that can be lost due to unexpected rate changes compared to the expected future value.

Investment Banking	Investment banking is the area of banking covering issues, trading, repos and corporate and take-over financing.
Liquidity Risk	The risk stemming from the lack of marketability of an investment that cannot be bought or sold quickly enough to prevent or minimise a loss.
Market Risk	The risk of adverse movements in market factors (such as asset prices, foreign exchange rates, interest rates) that cause volatility in P&L.
Operational Risk	The risk of loss resulting from inadequate or failed internal processes, people and systems or from external events.
Peak Over Threshold Method	This approach in EVT only considers the observations that exceed a predetermined threshold in modelling extreme events.
Private Banking	Private banking designates a service unit that provides securities safekeeping, investment advice and lending to their wealthiest customers.
Regulatory Capital	The capital buffer determined by the requirements prescribed by the Bank of International Settlement.
Relationship Banking	In relationship banking the client-bank relation is stable, and based on mutual confidence. By maintaining this relationship various services can be generated. Each transaction is evaluated individually.
Return Level	The level that is expected to be exceeded, on average, only once in a sequence of k periods of length n.
Sensitivity	A sensitivity or sensitivity indicator captures the deviation of a target variable due to the adverse change of a single parameter (the explanatory

	variable) by one unit.
Tail Conditional Expectation	*See* Expected Shortfall
Trading	Short-term purchasing and selling of securities with the goal of exploiting short-term market fluctuations.
Trading Book	The trading book groups all market transactions tradable in the market. Accounting rules for the trading book rely on market values (mark-to-market) of transactions and P&L, which are variations of the mark to market value between two dates.
Value At Risk	The maximum expected loss (measured in monetary units) of an asset value (or a portfolio) over a given time period and at a given level of confidence (or with a given level of probability), under normal market conditions.
Volatility	Volatility is the dispersion around the mean of any random parameter or target variable. It can be quantified by means of the standard deviation of the values of the target variable.

Bibliography

Acerbi, C. And Tasche D. (2001), *"Expected Shortfall: A Natural Coherent Alternative to Value at Risk"*, paper

Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1997), *"Thinking Coherently"*, RISK, Vol. 10, No. 11, November

Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1998), *"Coherent Measures Of Risk"*, paper

Bensalah, Y. (2000), *"Steps in Applying Extreme Value Theory to Finance: A Review"*, working paper, Bank of Canada, Canada

Besis, J. (2002), *"Risk Management in Banking"*, John Wiley & Sons Ltd, England

Bodie, Z. and Merton, R.C. (2000), *Finance (Chapters 10-13)*, Prentice Hall Inc., New Jersey

Coronado, M. (2000), *"Extreme Value Theory for Risk Managers: Pitfalls and Opportunities in the use of EVT in Measuring VaR"*, paper, University P. Commillas de Madrid, Spain

Embrechts, P, Klüppelberg, C. and Mikosch, T. (1997), *"Modelling Extremal Event: for insurance and finance"*, Springer, Berlin

Gilli, M. and Këllezi, E. (2003), *"An Application of Extreme Value Theory for Measuring Risk"*, preprint for Elsevier Science, University of Geneva and FAME, Switzerland

McNeil, A.J. (1999), *‘Extreme Value Theory for Risk Managers”* , Paper, ETH Zentrum, Zurich, Switzerland.

Oosterhof, prof. dr. J. and Vaart, prof. dr. A.W. van der (1999), *“Algemene Statistiek”* , lecture notes, Faculty Exact Sciences, vrije Universiteit amsterdam.

Appendix A. R Code for Practical Example

```
// Reading data from files
x <- read.table("C:/Mijn documenten/NAZIA/Werkstuk/Practical Example/SP500.txt")
xlog <- read.table("C:/Mijn documenten/NAZIA/Werkstuk/Practical Example/SP500log.txt")

l =dim(xlog)[1]
xloginv = xlog
for (i in 1:l)
xloginv[i,1]=xlog[i,1]*-1

// Plotting time series (figure 5.1 en 5.2)
par(mfrow=c(2,1), mgp = c(2,0.5,0), cex.lab = 0.8, cex.axis = 0.6, cex.main = 0.1, cex.sub = 0.8,
font.lab=3)
plot(x[,2],type="l", xlab = "Daily quotes of the S&P500 Index ",ylab="", col = rgb(0.4,0.4,0.6))
plot(xlog[,1],type="l", xlab = "Daily log return of the S&P500 Index ", ylab="",col =
rgb(0.4,0.4,0.6))

length = dim(xloginv)
stdev = sqrt(var(xloginv[,1]))
mu = mean(xloginv[,1])
u = seq(-0.03,0.04, by=0.00001)
threshold = mu+2*stdev
library(evir)

// Plotting Empirical vs. normal distribution (figure 5.3 en 5.4)
par(mfrow=c(2,2), mgp = c(2,0.5,0), cex.lab = 0.8, cex.axis = 0.6, cex.main = 1, cex.sub = 0.8,
font.lab=3)
hist(xloginv[,1], xlab = "Histogram of daily losses",ylab="",main="", col = "grey", freq =
FALSE,ylim = c(0, 110))
lines(u,dnorm(u,mu,stdev), col = rgb(0.4,0.4,0.6))
hist(xloginv[,1], xlab = "Histogram of daily losses",ylab="",main="", col = "grey", freq = TRUE, ,
ylim=c(0,5),xlim=c(threshold,0.040))
lines(u,dnorm(u,mu,stdev), col = rgb(0.4,0.4,0.6))
```

```
qqnorm(xloginv[,1], col = rgb(0.4,0.4,0.6))
meplot(xloginv[,1], col = rgb(0.4,0.4,0.6))


// Block Maxima Method
// Estimating Parameters of GEV
x1 = gev(xloginv[,1])
x5 = gev(xloginv[,1],block = 5)
x21 = gev(xloginv[,1],block = 21)
x63 = gev(xloginv[,1],block = 63)
x125 = gev(xloginv[,1],block = 125)
x250 = gev(xloginv[,1],block = 250)
plot(x1, col = "grey")
plot(x5, col = "grey")
plot(x21, col = "grey")
plot(x63, col = "grey")
plot(x125, col = "grey")
plot(x250, col = "grey")

// Table 5.1
xi1 = x1$par.ests[1]
xi5 = x5$par.ests[1]
xi21 = x21$par.ests[1]
xi63 = x63$par.ests[1]
xi125 = x125$par.ests[1]
xi250 = x250$par.ests[1]
sigma1 = x1$par.ests[2]
sigma5 = x5$par.ests[2]
sigma21 = x21$par.ests[2]
sigma63 = x63$par.ests[2]
sigma125 = x125$par.ests[2]
sigma250 = x250$par.ests[2]
mu1 = x1$par.ests[3]
mu5 = x5$par.ests[3]
mu21 = x21$par.ests[3]
mu63 = x63$par.ests[3]
mu125 = x125$par.ests[3]
mu250 = x250$par.ests[3]
sexi1 = x1$par.ses[1]
sexi5 = x5$par.ses[1]
sexi21 = x21$par.ses[1]
sexi63 = x63$par.ses[1]
sexi125 = x125$par.ses[1]
sexi250 = x250$par.ses[1]
sesigma1 = x1$par.ses[2]
sesigma5 = x5$par.ses[2]
sesigma21 = x21$par.ses[2]
sesigma63 = x63$par.ses[2]
```

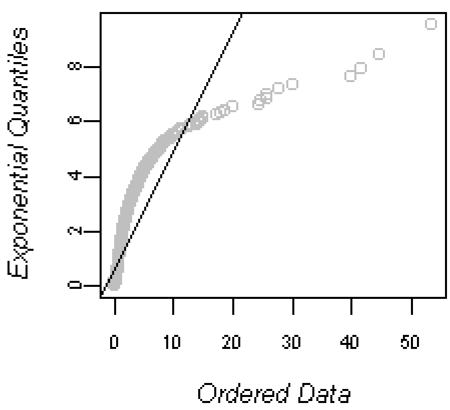
```
sesigma125 = x125$par.ses[2]
sesigma250 = x250$par.ses[2]
semu1 = x1$par.ses[3]
semu5 = x5$par.ses[3]
semu21 = x21$par.ses[3]
semu63 = x63$par.ses[3]
semu125 = x125$par.ses[3]
semu250 = x250$par.ses[3]

// Assessing 'Goodness of Fit'
// Figure 5.5
par(mfrow=c(1,2), mgp = c(2,0.5,0), cex.lab = 0.8, cex.axis = 0.6, cex.main = 0.1, cex.sub = 0.8, font.lab=3)
hist(xloginv[,1], xlab = "Histogram of daily losses",ylab="",main="", col = "grey", freq = FALSE, ylim=c(0,120))
lines(u,dgev(u,xi1,sigma1,mu1),col= rgb(0.7,0.7,0.8))
lines(u,dgev(u,xi5,sigma5,mu5),col= rgb(0.4,0.4,0.6))
lines(u,dgev(u,xi21,sigma21,mu21),col= rgb(0.29,0.29,0.44))
lines(u,dgev(u,xi63,sigma63,mu63),col= rgb(0.2,0.2,0.6))
lines(u,dgev(u,xi125,sigma125,mu125),col= rgb(0.15,0.15,0.23))
lines(u,dgev(u,xi250,sigma250,mu250),col= rgb(0.0,0.0,0.4))
hist(xloginv[,1], xlab = "Histogram of daily losses",ylab="",main="", col = "grey", freq = FALSE, ylim=c(0,2),xlim=c(threshold,0.040))
lines(u,dgev(u,xi1,sigma1,mu1),col= rgb(0.7,0.7,0.8))
lines(u,dgev(u,xi5,sigma5,mu5),col= rgb(0.4,0.4,0.6))
lines(u,dgev(u,xi21,sigma21,mu21),col= rgb(0.29,0.29,0.44))
lines(u,dgev(u,xi63,sigma63,mu63),col= rgb(0.2,0.2,0.6))
lines(u,dgev(u,xi125,sigma125,mu125),col= rgb(0.15,0.15,0.23))
lines(u,dgev(u,xi250,sigma250,mu250),col= rgb(0.0,0.0,0.4))

// Determining the Return Level
library(fExtremes)
fit = gevFit(xloginv[,1], block = 5, type = "mle", gumbel = FALSE)

// Table 5.2
gevrlevelPlot(fit, k.block = 4, main = "S&P 500: Return Levels")
gevrlevelPlot(fit, k.block = 13, main = "S&P 500: Return Levels")
gevrlevelPlot(fit, k.block = 26, main = "S&P 500: Return Levels")
gevrlevelPlot(fit, k.block = 52, main = "S&P 500: Return Levels")


// POT Method
library(evir)

// Figure 5.6
par(mfrow=c(1,2), mgp = c(2,0.5,0), cex.lab = 0.8, cex.axis = 0.6, cex.main = 0.1, cex.sub = 0.8, font.lab=3)
meplot(xloginv[,1], col="grey", xlim=c(0.005,0.012),ylim=c(0,0.005))
hill(xloginv[,1],start=15,end=125)
```

```
// Determining the threshold
t1 = findthresh(xloginv[,1],70)
t2 = findthresh(xloginv[,1],98)
t3 = findthresh(xloginv[,1],124)

// Parameter Estimation for GPD
library(fExtremes)

// Table 5.3
fit1 = gpdFit(xloginv[,1], nextremes = 70, type = "mle")
potxi1 = fit1$par.ests[1]
potbeta1 = fit1$par.ests[2]
potsexi1 = fit1$par.ses[1]
potsebeta1 = fit1$par.ses[2]
fit2 = gpdFit(xloginv[,1], nextremes = 98, type = "mle")
potxi2 = fit2$par.ests[1]
potbeta2 = fit2$par.ests[2]
potsexi2 = fit2$par.ses[1]
potsebeta2 = fit2$par.ses[2]
fit3 = gpdFit(xloginv[,1], nextremes = 124, type = "mle")
potxi3 = fit3$par.ests[1]
potbeta3 = fit3$par.ests[2]
potsexi3 = fit3$par.ses[1]
potsebeta3 = fit3$par.ses[2]

// Assessing 'Goodness of Fit'
// Figure 5.7
par(mfrow=c(1,2), mgp = c(2,0.5,0), cex.lab = 0.8, cex.axis = 0.6, cex.main = 0.1, cex.sub = 0.8,
font.lab=3)
hist(xloginv[,1], xlab = "Histogram of daily losses",ylab="",main="", col = "grey", freq = FALSE,
ylim=c(0,2),xlim=c(threshold,0.040))
lines(u,dgpd(u,xi=potxi1,beta=potbeta1),col= rgb(0.7,0.7,0.8))
lines(u, dgpd(u,xi=potxi2,beta=potbeta2), col= rgb(0.29,0.29,0.44))
lines(u,dgpd(u,xi = potxi3, beta=potbeta3),col= rgb(0.0,0.0,0.4))
hist(xloginv[,1], xlab = "Histogram of daily losses",ylab="",main="", col = "grey", freq = FALSE,
ylim=c(0,0.05),xlim=c(threshold,0.040))
lines(u,dgpd(u,xi=potxi1,beta=potbeta1),col= rgb(0.7,0.7,0.8))
lines(u, dgpd(u,xi=potxi2,beta=potbeta2), col= rgb(0.29,0.29,0.44))
lines(u,dgpd(u,xi = potxi3, beta=potbeta3),col= rgb(0.0,0.0,0.4))

// Determining VaR and ES
library(evir)
distr = gpd(xloginv[,1], threshold = t3, method = "ml")
plotdistr = plot(distr)

// Table 5.4
gpd.q(plotdistr, 0.925, ci.type = "likelihood", ci.p = 0.95, like.num = 50)
gpd.q(plotdistr, 0.95, ci.type = "likelihood", ci.p = 0.95, like.num = 50)
gpd.q(plotdistr, 0.975, ci.type = "likelihood", ci.p = 0.95, like.num = 50)
```

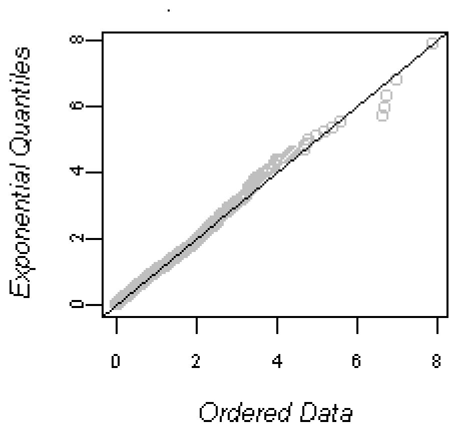
```
gpd.q(plotdistr, 0.99, ci.type = "likelihood", ci.p = 0.95, like.num = 50)
gpd.q(plotdistr, 0.999, ci.type = "likelihood", ci.p = 0.95, like.num = 50)
gpd.q(plotdistr, 0.9999, ci.type = "likelihood", ci.p = 0.95, like.num = 50)

gpd.sfall(plotdistr, 0.925, ci.p = 0.95, like.num = 50)
gpd.sfall(plotdistr, 0.95, ci.p = 0.95, like.num = 50)
gpd.sfall(plotdistr, 0.975, ci.p = 0.95, like.num = 50)
gpd.sfall(plotdistr, 0.99, ci.p = 0.95, like.num = 50)
gpd.sfall(plotdistr, 0.999, ci.p = 0.95, like.num = 50)
gpd.sfall(plotdistr, 0.9999, ci.p = 0.95, like.num = 50)
```

Appendix B. QQ plots for Residuals

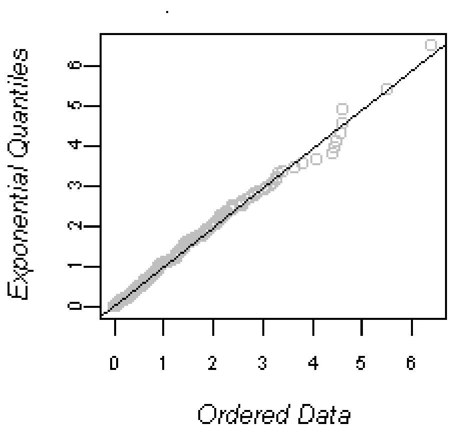
Daily Returns



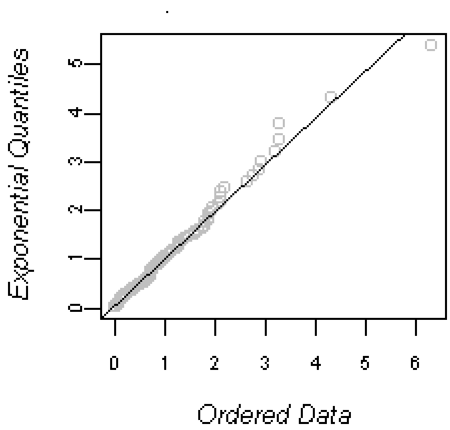
Weekly Returns



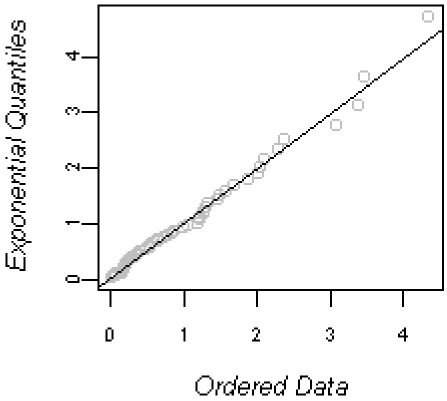
Monthly Returns



Quarterly Returns



Returns per Semester



Yearly Returns