

# The Spiral-Down Effect in Revenue Management



BMI – Thesis

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## **Preface**

The BMI thesis is one of the final compulsory parts within the Masters program Business Mathematics & Informatics. The goal is to analyze and describe a subject in the field of BMI based on existing literature.

In April 2009 Dr. Ger Koole introduced me to the “The spiral-down effect in revenue management”, the subject of this paper, based on an article written by Cooper et al (2006). Their paper gave rise to the literature research I have done on this subject. In addition, I was very much interested in finding ways to prevent the spiral-down effect. Therefore I have looked for ways of doing so, and researched a solution that from my point of view had the potential of solving the problem.

This thesis was written during the months of June and August 2009 under supervision of Dr. Elena Tzenova. I would like to thank her for the guidance and support in the process of writing my thesis.

## **Management Summary**

The subject of revenue management has been widely discussed in literature. It focuses on maximizing revenue, by altering the availability of a product or service throughout a number of price-classes. A revenue manager iteratively goes through a three-step process. First he determines a so-called 'protection level' for each price class. This can be interpreted as the number of units that are reserved for customers who are willing to buy in that price class. In the second step these protection levels are being used while selling the product or service. Based upon these events, the revenue manager can collect data considering the demand in each price class. In the third step, the revenue manager forecasts the demand in the next period the product or service is sold, based on the data he collected. He can now go back to step one, in which he will use those forecasts to set new protection levels.

Most researchers in the field of revenue management have concentrated on the first step of the process described above. Given a demand forecast, they determined protection levels leading to the maximum revenue. Famous methods that have resulted from their studies are the Littlewood rule and the EMSRb. These methods have been applied in a variety of businesses that face revenue management decisions.

However, in practice there are two main differences. The underlying assumptions of the optimization method do not always hold and the method is systematically being used and updated. Methods like EMSRb and Littlewood for instance, assume exogenous demand throughout price classes. This means that the demand in each price class is independent of the demand in any other price class. In practical situations, this does not always hold. Often there are customers who are willing to pay more money if cheaper price classes are sold out. When applying the method only once under such erroneous assumptions, reasonable results can still be obtained. However, it is in the long run application of the optimization method where a 'spiral-down effect' can emerge, in which controls and revenue systematically decreases.

An illustrative example of the spiral-down effect is when we consider customers that are willing to pay any price, but buy in the cheapest price class that is available. Assuming exogenous demand throughout price classes, the revenue manager observes the number sales in each price class as its corresponding demand. While observing a high demand in cheaper price classes, he might decide to create more availability in those price classes. This leads to even more sales in the cheaper classes and less in the more expensive ones. This spiral-down can continue until the revenue manager sells his product at the lowest price, even though every customer is willing to pay the highest price.

Cooper et al (2006) suggested some ways of preventing the spiral-down effect. One of these is using an extended version of the EMSRb method, which incorporates customer diversion. Using simulations for different settings, this method proved to make the problem less. However it did not completely get rid of the spiral-down. Additionally it should be mentioned that preventing the spiral-down effect should never be the main focus. Some methods might prevent a spiral-down, but perform poorly as it comes to maximizing the revenue.

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# 1 Introduction

## 1.1 Revenue management

The subject of revenue management has been widely discussed in scientific literature. It involves maximizing the revenue or profit for a perishable resource by controlling its prices and availabilities. Here, a resource is called perishable when it has no value after a certain amount of time. This is usually the case with a lot of services that are being offered. After the moment of execution, the service cannot be sold anymore therefore no longer has a value. Therefore the goal of revenue management is to generate as much revenue as possible before the resource is perished.

A popular field of application of revenue management is the airline industry. Here tickets for flight itineraries are sold at different prices through time in order to gain maximum revenue from a set of flights. As soon as the flight takes off, resources in the form of empty seats cannot be sold anymore and are therefore perishable. In the airline industry they often refer to the term 'yield management'. The yield is defined as the revenue an airline makes per passenger per unit of distance.

The internet and introduction of electronic commerce have stimulated the applications of revenue management. Using those instruments it became a lot easier for an organization to manage and sell large amounts of products at different prices over time. The optimal values for these amounts and prices were then determined by combining forecasting techniques and mathematical optimization methods into automated systems.

The first mathematical models on revenue management were published by Littlewood (1972). The outcome of his analysis is known as the Littlewood-rule and uses the marginal revenue of a product to determine its availabilities in two different price-classes.

Later research by Belobaba (1989) resulted in a model that was also applicable for more than two price-classes. His model has been the foundation for much research that has been done after that on revenue management. Examples are models to compute the optimal controls for a set of products that are in some way related to each other, instead of only a single product.

This thesis will touch upon a specific problem that can occur in some revenue management models that have been studied for years and are currently used in a large number of organizations.

## 1.2 Objective

Most researchers in the area of revenue management have focused on developing models that aim at determining availabilities of a product in different price classes over time. Under a number of assumptions they show that their models can find the controls that lead to a maximum revenue. Examples of such models are the Littlewood rule and EMSRb. In most revenue management applications however, the underlying assumptions of the model do not always apply and the model is used multiple times.

Recent work of William L. Cooper, Tito Homem-de-Mello and Anton J. Keywegt (2006) explained how models like the Littlewood rule, can lead to the so-called ‘Spiral-down effect’ on the long term. They showed that under certain circumstances there will be a downward spiral on the controls and the revenue itself. The focus in their work is not just on one single application of the model, but on the long term process of consequently applying and updating the optimization methods.

Their article also addresses the fact that revenue managers in the field are usually not aware of this phenomenon. A reason for this could be that they have too little knowledge about the underlying techniques. Even if they know what assumptions these techniques are based on, they might not know exactly what the consequences are if these assumptions do not apply. One of these consequences could be systematically worsening controls and revenue.

For this reason it would be interesting to have a good overview of the results from the research mentioned above and other related work. Even more interesting would be to go a bit further and look for ways of dealing with this problem. Therefore, the main research question addressed in this thesis is as following:

“What is the spiral-down effect in revenue management and how can it be prevented?”

Evidently, this can be divided into two sub questions:

*“What is the spiral-down effect and what would be a reason for it?”*

There are explicit conditions under which the spiral-down effect occurs. It is therefore important to analyze what these conditions are and how they lead to a downward spiral on the revenue. This question can be answered using the results from the article mentioned above.

*“How can we prevent the spiral-down effect?”*

Up to this point, there has not been much research done in methods that prevent the spiral-down effect from happening. This is why it can be of great value to try alternative models and see if the spiral-down occurs. In the paper by Cooper et al (2006) the writers give some suggestions on what kind of models could be used. An answer to this question will be given by analyzing the suggested models putting them to the test in a simulation.

### **1.3 Outline of this thesis**

The rest of this thesis is organized as follows. In chapter two the first sub question will be answered. To do so, we will first present the process a revenue manager repeatedly goes through. With that knowledge in hand, the idea behind the spiral-down effect can be explained. To illustrate the problem, we will also give a deterministic example. Then we will look into some specific cases of forecasting techniques: the affine updates and the empirical distribution methods. We will derive the exact circumstances under which a spiral-down effect occurs for those methods. In the end of chapter two we will show the spiral-down effect for the optimization technique EMSRb, by means of simulation.

In chapter three the second research question will be addressed. It is of our interest in this chapter that we want to study an alternative optimization technique, which will be an extension of EMSRb incorporating customer diversion. First we will explain how this model works. Then we will show the results of a number of simulations, where the model is applied on the same setting as in chapter two.

In the chapters four we will draw conclusions based upon the results obtained throughout the whole thesis. Since not much research has been done on the problem discussed in this thesis, we will also point out a number of directions for further research.

## 2 The spiral-down effect

In this chapter we will be answering the first research question: *“What is the spiral-down effect and what would be a reason for it?”* The answer can be given by using the results from the paper by Cooper et al (2006). Models such as the Littlewood rule usually have a number of assumptions regarding an underlying stochastic or deterministic demand process. If those assumptions apply, an optimal policy is derived. In the airline industry a typical example of such a policy could be a proper seat allocation among numerous fare classes. However, it is in those practical situations that the model is used in a context where the assumptions do not apply.

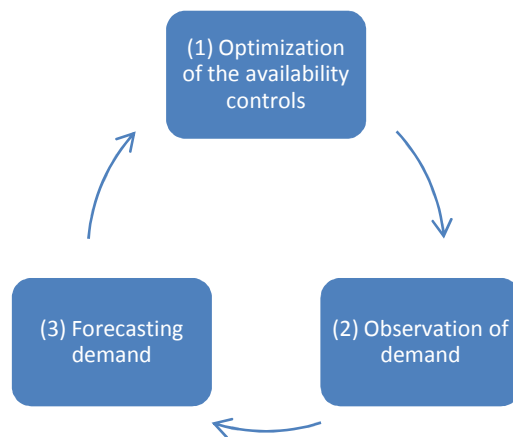
Cooper et al (2006) analyzed a long-term process in which the Littlewood rule was continuously being used and updated to determine the revenue management controls. They showed that when underlying assumptions of the Littlewood rule fail, a so-called ‘spiral-down effect’ can emerge on the controls and the revenue. This means that at every step of the iterative optimization process, the optimality of its solution decreases. Such a phenomenon typically occurs when in each of those iterations the quality of the models’ solution depends on the solution of its predecessor.

In this chapter we will first analyze the process in which a revenue manager continuously forecasts demand and optimizes controls. Using that knowledge we can explain why the spiral-down occurs. Then we will analyze some forecasting techniques which are considered to be ‘good’ and show under which circumstances they will lead to a spiral-down. In the end of this chapter we will simulate the effect with a model widely used in practice.

### 2.1 The revenue management forecasting and optimization process

In order to understand the problem, it is important to get an idea of how the forecasting and optimization process in revenue management works. In this section we will elaborate on this process, which is summarized in Figure 2.1. At the start of this process a revenue manager needs to have some idea of what the demand for his service will be like. Then, using a mathematical optimization method he can find the optimal policy for selling this service (1). While the policy is being applied and the service is being sold, new information about the demand is observed (2). Combining this new data and the history of the demand that was gained earlier, a new demand forecast can be made (3). We are now back where we started, since this forecast is used in the first step to derive an optimal policy.





*Figure 2.1: The illustrated cycle in the forecasting and revenue management process*

### **2.1.1 Optimization of the availability controls**

The setting of a revenue management problem is usually a limited amount of a service that is sold, for instance seats on a flight. This service is sold in a number of pre-determined price classes.

Through time customers arrive and buy the service if it is still available in a price class they desire. After a certain amount of time the service is executed and leftovers will always remain unsold.

The decision variable in this context is the amount of services that are made available in each price class. In the case studied in this paper, these price classes are nested. This means that demand for a more expensive price class can have access to unused availability in cheaper price classes, but not the other way around. This ensures that a high-fare request is never turned away as long as the service is still available. The goal in such a nested model is to find the ‘protection levels’. A protection level is the number of services specifically reserved for a fare class and all higher classes. This means that from the moment the leftover availability reaches this protection level, lower classes are no more admitted for purchasing the service.

Evidently, these protection levels need to be determined in such a way that the revenue is maximized. If they are set too low, high fare customers could be turned away because too much capacity was sold at a lower fare. If they are set too high, a number of services might remain unsold as they were reserved for higher fare passengers, but these passengers never showed up. In both cases, the revenue can be increased by altering the protection levels. This should be done in such a way that the higher fare customers are always served and leftover capacity is kept to a minimum by filling it up with lower fare customers.

A well-known model for solving the 2-class version of this allocation problem is the Littlewood rule, proposed in 1972 by Littlewood. This method uses the probability distribution of the assumed exogenous demand for the high fare class and its marginal revenue to choose the optimal protection level. We will now give the formulation of this method.

In this setting there are two fare classes, class 1 and class 2. The fare for class 1 is denoted by  $f_1$  and the fare for class 2 is denoted by  $f_2$ . It is assumed that class 1 is more expensive than class 2, so  $f_1 > f_2$ . Let  $H$  denote the cumulative probability distribution for the exogenous class-1 demand. Then the revenue manager will choose a protection level  $l$  that satisfies:

$l \in H^{-1}(\gamma)$ , where  $\gamma := 1 - f_2/f_1$  and  $H^{-1}(\gamma)$  denotes the set of  $\gamma$ -quantiles of  $H$ .

This means that the lowest  $l$  is chosen for which applies:  $H(l) \geq \gamma$ . In the special case that the demand distribution is continuous, we choose the protection level that satisfies:

$$f_1 * \text{Prob}[\text{exogenous demand for class 1} \geq l] = f_2$$

As the protection level is determined according to this rule, the customer admission policy will be as follows. As long as the availability of the service is more than  $l$ , allow all customers. Otherwise, allow only class-1 customers. Of course, none of the customers are accepted when there is no more availability left over.

In some articles, the term ‘booking limit’ is used instead of protection level. A booking limit is the maximum amount of the service that is available to customers of a certain fare-class. Now let  $c$  denote the total amount of the service that can be sold, then the booking limit for class 2 is equal to  $c - l$ . Furthermore, the booking limit for class 1 is  $c$ .

In practice, organizations divide the time in which they sell their services into multiple booking periods. At the beginning of each booking period the sales that have occurred up until that point are being examined and the protection levels are adjusted. For sake of simplicity, the cases studied throughout this thesis assume only one booking period. This means that the fixed protection levels are used until the service is executed.

### 2.1.2 Observation of demand

Let us elaborate on the two fare class example from the previous section. The revenue manager has determined the protection level for the single booking period and uses this to protect class-1 demand while selling the service. During this booking period customers come and go and new demand data can be observed. In practical situations however, not every arriving customer is observed. Take for example the situation where someone wants to buy a plane ticket in a certain fare class. When this person checks the availability on the carriers’ website, it appears that the tickets in the fare class he desired are sold out. This customer will not buy the plane ticket he wanted and is therefore not observed. This type of data, in which turned away customers are not recorded, is what we call ‘Truncated demand’. The opposite of this, ‘Untruncated demand’, would then be the idealistic situation where every arriving customer is recorded by the revenue manager.

The revenue manager eventually needs to estimate the distribution  $H$  used in the Littlewood rule. So he observes a quantity  $X$  assuming that it is exogenous class-1 demand. Since the Littlewood rule does not require any input for class-2 demand, this is the only statistic that matters. Also, let  $G(l, \cdot)$

be the cumulative distribution function of the observed quantity  $X$ , when using protection level  $l$ . It is important to realize that this distribution of  $X$  depends on  $l$ , since the choice of  $l$  influences the observed class-1 demand. However the real distribution of class-1 demand  $H$ , does not depend on  $l$ . This is because the desire for the customers to buy the service will be there, regardless its availability.

The following scenario is set-up according to the analysis done in Cooper et al (2006). Suppose there are three types of customers that are interested in buying the service, denoted with type-a, type-b and type-ab. Type-a customers are only interested in class-1; type-b customers will only buy class-2 and type-ab customers are willing to buy in both of the classes, but will always buy the cheapest one that is available. Since  $f_1 > f_2$  type-ab customers will buy class-2 if the availability is more than  $l$ . They will buy class-1 when the availability is at least 1 and less than  $l$ . In the case that the service is no more available, they will not buy at all.

These customers arrive according to a marked point process, without any kind of order. The process starts as the service is made available for sale and ends as it is being executed. The length of this interval is equal to the length of the booking period. As we have mentioned earlier, customers arrive whether or not there is availability. This means that the point process is independent of  $l$ .

Now let  $D_a$  and  $D_{ab}$  denote the number of type-a customers and the number of type-ab customers respectively that arrive in the arrival process. Also,  $D_a(l)$  and  $D_{ab}(l)$  are the number type-a and type-ab customers respectively that arrive until  $c - l$  tickets are sold. Then if  $l \geq c$ ,  $D_a(l) = D_{ab}(l) = 0$  and if the total demand is less than  $c - l$ ,  $D_a(l) = D_a$  and  $D_{ab}(l) = D_{ab}$ .

We will now present an expression for  $G(l, \cdot)$  following Cooper et al (2006). However, the distribution function  $G(l, \cdot)$  in the case of truncated demand is different from the case of untruncated demand, since the observed numbers are different. Therefore, we will handle both cases separately.

- **Untruncated demand**

In this case, we do not have to worry about customers that arrive after all tickets are sold, since every customer arrival is recorded. Therefore the class-1 demand that is observed by the revenue managers is equal to the number of type-a customers plus the number of type-ab customers that arrive after  $c - l$  services are sold. So  $X = D_a + D_{ab} - D_{ab}(l)$ . Using the fact that  $G(l, \cdot)$  is the cumulative distribution function of  $X$ ,  $G(l, x) = Prob[X \leq x] = Prob[D_a + D_{ab} - D_{ab}(l) \leq x]$ . Note that type-ab customers that arrive after all services are sold are assumed to be observed as class-1 demand.

- **Truncated demand**

Now the revenue manager stops recording customer arrivals after selling  $c$  tickets. The observed class-1 demand can be split into two parts:

- The number of type-a customers arriving before  $c - l$  services are sold, which is:  $D_a(l)$ .
- The number of type-a and type-ab customers arriving after  $c - l$  services are sold, which is:  $D_a - D_a(l) + D_{ab} - D_{ab}(l)$ . However, the revenue manager stops recording when there is no more availability. This is when the number of arriving

type-a and type-b customers after selling  $c - l$  services reaches  $l$  if  $l < c$  or when it reaches  $c$  if  $l \geq c$ . This means we get the following:

$$\min\{D_a - D_a(l) + D_{ab} - D_{ab}(l), \min\{c, l\}\}$$

We can rewrite this as:

$$= \min\{D_a - D_a(l) + D_{ab} - D_{ab}(l), c, l\}$$

Combining these two parts gives:

$$X = D_a(l) + \min\{D_a - D_a(l) + D_{ab} - D_{ab}(l), c, l\}$$

And thus:

$$G(l, x) = \text{Prob}[D_a(l) + \min\{D_a - D_a(l) + D_{ab} - D_{ab}(l), c, l\} \leq x]$$

Note that the distribution  $G$  depends on the protection level  $l$  whether we have truncated or untruncated demand data.

### 2.1.3 Forecasting demand

Up to this point the revenue manager has computed a protection level ( $l$ ) to protect class-1 demand and observed the assumed exogenous demand for class-1 ( $X$ ). Based upon how this demand is observed (truncated or untruncated), we know the structure of its cumulative probability distribution ( $G$ ).

The next step in this process is to make an estimate of the actual distribution of the class-1 demand ( $H$ ). This is often done using a forecasting technique. However, the outcome of these methods is usually not only based on the last observation, but on the complete history of the observed demand. Now let one period for the revenue manager be the same as going through the cycle once in Figure 2.1. During period  $k$  the revenue manager uses protection level  $L^{k-1}$  and observes demand quantity  $X^k$ . We assume that with probability 1, for each  $k \in \mathbb{N}$ ,

$$\mathbb{P}(X^k \leq x | \mathcal{F}^{k-1}) = G(L^{k-1}, x)$$

This means that the conditional probability distribution of  $X^k$  given its history up to period  $k - 1$ , only depends on  $L^{k-1}$ .

Let  $\hat{H}^0$  be the initial estimation for the probability distribution of class-1 demand at the beginning of period 1. Then at the end of period  $k$  the estimation of  $H$  is done according to:

$\hat{H}^k := \phi^k(\hat{H}^0, X^1, \dots, X^k)$ , where  $\phi^k$  is determined by the forecasting technique the revenue manager chooses to use. The forecasting techniques that are considered in this thesis are the ones that are good in a certain sense. Following Cooper et al (2006), by ‘good’ we mean that if the distributions  $G(L^k, \cdot)$  settle down as  $k$  gets large, then the estimations  $\hat{H}^k$  will approach the same limit. In section 2.3 we will analyze some models that are considered ‘good’ forecasting techniques.

Once the estimation  $\hat{H}^k$  is determined, we are back at the start of the cycle. The protection level can then be determined for the next period using the Littlewood rule:

$$L^k \in \hat{H}^{-1}(\gamma)$$

This protection level can then be used in period  $k + 1$  to protect class-1 demand.

## 2.2 Systematically worsening controls and revenue

In the previous section we have shown how the iterative process works of optimizing policies, observing and forecasting demand. Now we can examine how in this process the spiral-down effect evolves which leads to systematically worsening controls and revenue. This will be done by elaborating further on the two fare-class example, with a deterministic demand. The example is performed in a similar fashion as the example given in Cooper et al (2006), however later on we will be using different parameters values.

Assume that the overall demand is deterministic and equal to  $d$ . Moreover, all of the customers are of the type ab. This means that every arriving customer is willing to pay for class 1, but will buy in class 2 if it is available. A complex mathematical model determining the protection level is not needed in this situation. The revenue would be maximized by setting the protection level equal to the capacity (or higher), since we can make every customer pay the higher class-1 price. The revenue manager however, does not know this and will determine the protection level according to what he observes as class-1 demand.

Now if we assume that the data collected by the revenue manager is untruncated, we can use the derived formulas for untruncated demand from section 2.1.2. There we saw that:

$$X = D_a + D_{ab} - D_{ab}(l).$$

We only have customers of type ab, which means that  $D_a = 0$  and  $D_{ab} = d$ . If the demand is less than  $c - l$ , then  $D_{ab}(l)$  is equal to  $D_{ab}$ . Otherwise  $D_{ab}(l)$  is equal to  $(c - l)^+$ . Putting this together gives:

$$D_{ab}(l) = \min(d, (c - l)^+)$$

and hence using the formula for the observed demand  $X$ :

$$X = d - \min(d, (c - l)^+) = [d - (c - l)^+]^+$$

Which is now deterministic. Therefore we obtain directly its cumulative distribution function:

$$G(l, x) = \begin{cases} 1, & \text{if } x \geq [d - (c - l)^+]^+ \\ 0, & \text{otherwise} \end{cases}$$

We had already mentioned that the best value for  $l$  is the one that makes every arriving customer pay the class-1 fare. This happens when we take  $l \geq c$ . The worst value for  $l$  would be the one that

allows every arriving customer to buy in the cheaper class 2. That situation is created by setting  $l \leq (c - d)^+$ .

Suppose that the revenue manager starts with a protection level  $L^0 \geq 0$ . After each period  $k$  he uses the empirical distribution of the observed demand in periods 1 ...  $k$  to forecast the demand in period  $k + 1$ . This means that the estimated distribution function of the class-1 demand is defined as:

$$\hat{H}^k(x) := \frac{1}{k} \sum_{j=1}^k 1_{\{X^j \leq x\}} \quad \text{for every } k \in \mathbb{N}$$

We will now analyze three separate cases: (1)  $d = c$ , (2)  $d < c$ , (3)  $d > c$ .

- (1)  $d = c$ . In this case, the demand is equal to the capacity in every period. This means that the observed demand for class 1 is equal to the protection level. So for any protection level  $L^0 \in [0, c]$ ,  $X^1 = L^0$  and hence:

$$G(L^0, x) = 1_{\{L^0 \leq x\}}$$

$L^1$  is chosen according to the  $\gamma$ -quantile of the empirical distribution of the only observed value  $X^1$ . Thus, it follows that:  $L^1 = X^1 = L^0$ . Moreover,  $X^2 = L^1$  and since both observations incorporated in the empirical distribution are equal,  $L^2 = X^2 = L^0$ . This will go on for every following period, so we can state that  $L^k = L^0$  and  $\hat{H}^k = \hat{H}^1 = 1_{\{L^0 \leq x\}}$  for every  $k \in \mathbb{N}$ .

If  $L^0$  is chosen greater than  $c$ , the first observed demand value  $X^1$  will be exactly  $c$ . This means that  $L^1 = c$  and therefore  $X^2 = c$ . This process will keep going on in a similar fashion, thus:  $L^k = L^1 = c$ .

We can see that in this case when  $L^0 \in [0, c]$  the protection level remains the same throughout all periods. When  $L^0 > c$  the protection level goes down to  $c$  in the first period and then remains unchanged for all following periods. This means that for  $d = c$  there is no spiral down on the protection level and the revenue.

- (2)  $d < c$ . The demand for this case falls short to the total capacity, which causes leftovers at the end of every period. Earlier we derived that  $X$  is equal to  $[d - (c - L)^+]^+$ . So when using the initial protection level  $L^0$ ,  $X^1 = [d - (c - L^0)^+]^+$ . This formula is illustrated in Figure 2.2. We can observe from this that when  $d < c$ ,  $X^1$  will always be less than or equal to  $L^0$ . Furthermore, we can see that  $X^1 < L^0$  for  $L^0 \neq 0$ .

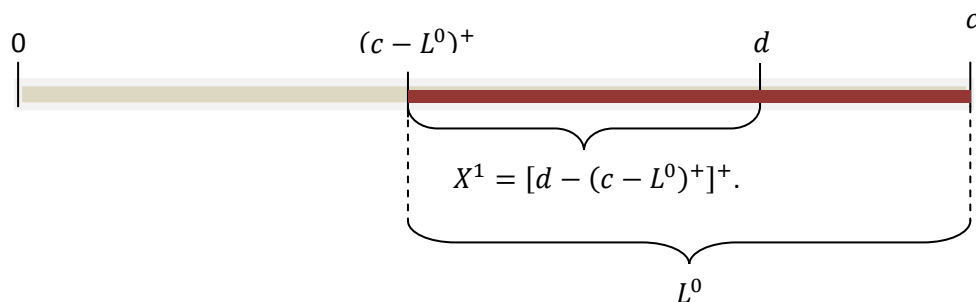


Figure 2.2: The formula for  $X$  illustrated.

We are using the empirical distribution to obtain the next protection level. Since  $X^1$  is the only observed value it will always hold that  $L^1 = X^1$ . Thus:  $L^1 \leq L^0$ .

For the observed class-1 demand in period two, we can again use the Figure 2.2 to show that:  $X^2 \leq L^1$ . Depending on the value of  $\gamma$ , the  $L^2$  will be either  $X^1$  or  $X^2$ . Since  $X^1 = L^1$  and  $X^2 \leq L^1$ ,  $L^2 \leq L^1$ . However, eventually after a finite number of periods the empirical distribution of the observed values will be such that its  $\gamma$ -quantile will be equal to  $X^2$ . More analysis on the empirical distribution will be done later in section 2.3. Proposition 1 from Cooper et al (2006) results as follows. In general it holds that  $L^k \leq L^{k-1}$  for every  $k \in \mathbb{N}$  and  $L^k > 0$ . Furthermore there exists a  $k^*$  such that  $L^j = 0$  for every  $j \geq k^*$ . This comes from the fact that once the protection level reaches 0 at period  $k^*$ , no more class-1 customers will be observed. Therefore a protection level greater than 0 will not be chosen after period  $k^*$ .

The analysis above clearly signals the spiral-down effect. We can also see that it takes the protection level to the worst possible value, where all customers that arrive buy a class-2 ticket and the revenue reaches a minimum.

It is also interesting to note that the revenue manager's estimates  $\hat{H}^k$  converge to point mass at zero. This is consistent with what the revenue manager is observing; namely that  $X^j = 0$  for a  $j$  large enough.

Based upon this analysis we can also point out the two factors that cause the spiral-down effect:

(1) *The use of an incorrect model.*

The model assumes the demand for class 1 to be exogenous. This is not the case since it depends on the demand in class 2 as well as the protection level. This has to do with the fact that customers are willing to pay for both classes. The Littlewood rule however, was designed for the situation where customers are only interested in a single fare-class.

(2) *Consequently using and updating the model.*

The spiral-down effect is a phenomenon that happens on the long term. It only occurs when the model is updated according to what is observed in previous periods in which the model was used.

Furthermore we can note that the problem is not caused by the fact that the revenue manager cannot observe every customer, since in this example we assumed untruncated demand data.

To illustrate how this problem develops through a number of time periods, we will give an example. This is done in the same way as in Cooper et al (2006), but with different parameter values.

## The Spiral-Down Effect in Revenue Management

Thom Eijken, July 09

In this example, we give the revenue manager the ‘benefit of the doubt’ by assuming that at the start of the first period he chooses the best possible protection level.

Furthermore the parameters were chosen in such a way that the results can be computed by hand. Let  $c = 60$ , then  $L^0 = 60$ . Moreover we take  $d = 50$ ,  $f_1 = 400$  and  $f_2 = 100$ . This means that  $\gamma = 1 - f_2/f_1 = 0.75$ . Recall that every arriving customer is willing to buy class 1, but will buy a class 2 if it is available. The resulting optimization and forecasting process can be viewed in Table 2.1.

k	Protection level $L^{k-1}$	$X^k$	Class-1 sales	Class-2 sales	Revenue
1	60	50	50	0	20000
2	50	40	40	10	17000
3	50	40	40	10	17000
4	50	40	40	10	17000
5	40	30	30	20	14000
...	...	...	...	...	...
16	40	30	30	20	14000
17	30	20	20	30	11000
...	...	...	...	...	...
64	30	20	20	30	11000
65	20	10	10	40	8000
...	...	...	...	...	...
256	20	10	10	40	8000
257	10	0	0	50	5000
...	...	...	...	...	...
1024	10	0	0	50	5000
1025	0	0	0	50	5000

Table 2.1: The spiral down effect illustrated.

We can see from this table that starting with the optimal protection level 60, it goes all the way down to 0 in 1025 periods. This is reflected in the revenue, which decreases with 75 %. This decrease has already been reached in 257 periods. Since all arriving customers are then able to buy the cheapest class, the revenue has reached a minimum. It can be noticed that when the protection level decreases, it jumps down with step-size 10. This can be explained by the fact that the empirical distribution of  $X$  was used as  $\hat{H}$ . Any quantile from this distribution will be one of the observed values. Since the observed demand decreases with step-size 10, the protection level will therefore also go down by 10.

- (3)  $d > c$ . In this case there is always more demand than there is capacity. If the initial protection level is greater than the capacity, then every arriving customer is observed as class-1 demand. This means that  $X^1 = d$ . Using the empirical distribution, the revenue manager will choose the next protection level equal to  $X^1$ . In the second period we face the same situation. Since it holds that  $L^1 > c$ , every customer is still forced to buy class 1.



Thus also here we obtain:  $X^2 = d$ . It follows from this that for every period  $k \geq 1$ ,  $X^k = L^k = d$ .

If the initial protection level is less than the capacity,  $X^1 = L^0 + d - c > L^0$ . The new protection level  $L^1$  will then be equal to  $X^1$ , so  $L^1 > L^0$ . If this protection level is less than the demand,  $X^2 > L^1$ . After a number of periods, this will force the protection level to be equal to  $X^2$ . This process will go on until the protection level is equal to the demand. From this, proposition 2 from Coopers et al (2006) follows: For a given  $L^0 \in [0, c]$ , we can state that  $L^{k+1} \geq L^k$  for all  $k$ . Furthermore there exists a  $k^\circ$  such that  $L^j = d$  and  $X^j = d$  for all  $j \geq k^\circ$ .

We can conclude that in the case where  $d > c$  there is an upward spiral in the protection level, which eventually converges to  $d$ .

In the cases explained above, the spiral-down only occurs when there is leftover availability at the end of the period. In general however, this is not always the case. Such a condition depends mostly on the forecasting technique that is used. In the next section we will therefore derive these conditions for three specific techniques.

## 2.3 Forecasting methods

As we could see in the previous section, the way in which the protection level behaves through time is for some part dependent on the used forecasting method. Therefore we will analyze two forecasting techniques in this chapter to find out under what circumstances the spiral-down effect occurs. The first method will be the “Affine updates”, which can be used in combination with different probability distributions. Here we will use the normal distribution and the exponential distribution. The second method is the “Empirical distribution”. We have already used this for the example in the previous section. Here we will look at a more general situation.

The analysis and its results in this section is similar to that in Cooper et al (2006).

### 2.3.1 Affine updates

Assume that the sequence of protection levels using the affine updates method satisfies the following inductive equation:

$$\mathbb{E}(L^k | \mathcal{F}^{k-1}) = L^{k-1} + \frac{1}{k} [\alpha - (1 - \beta)L^{k-1}] = \frac{k-1+\beta}{k} L^{k-1} + \frac{\alpha}{k}.$$

This is the expectation of the protection level determined at the end of period  $k$ , given the history before period  $k$ . Cooper et al (2006) rewrite this as follows:

$$\mathbb{E}(f^k L^k - g^k | \mathcal{F}^{k-1}) = f^{k-1} L^{k-1} - g^{k-1}$$

and consider the sequence  $\{f^k, g^k\}_{k=m}^\infty$  where:

$$f^{m-1} := 1, \quad f^k := f^{k-1} \frac{k}{k-1+\beta} \quad \text{for all } k \geq m,$$

$$g^{m-1} := 0, \quad g^k := g^{k-1} + \frac{\alpha}{k} f^k \quad \text{for all } k \geq m.$$

Here  $m$  is taken to be a positive integer such that  $m - 1 + \beta > 0$ .

Suppose the following assumption holds:

*Assumption A: The sequence  $\{f^k, g^k, L^k\}_{k=m}^{\infty}$  satisfies  $\sup_{k \geq m} \mathbb{E}[f^k L^k - g^k] < \infty$ .*

Then the sequence  $\{f^k L^k - g^k\}$  forms a convergent martingale. This means that the following holds:  $\mathbb{E}(f^k L^k - g^k | \mathcal{F}^{k-1}) = f^{k-1} L^{k-1} - g^{k-1}$  and there exists a random variable  $Y$  such that  $f^k L^k - g^k \rightarrow Y$  with probability 1 as  $k$  goes to infinity. Moreover we can state the following about the behavior of the protection level, as given in Cooper et al (2006):

1. If  $\beta < 1$  and  $\alpha = 0$ , then  $L^k \rightarrow 0$  with probability 1.
2. If  $\beta = 1$  and  $\alpha = 0$ , then  $\{L^k\}$  is a martingale and  $L^k \rightarrow Y$  with probability 1.
3. If  $\beta = 1$  and  $\alpha \neq 0$ , then  $L^k \rightarrow \text{sgn}(\alpha) * \infty$  with probability 1. This means that in the case  $\alpha < 0$ , a spiral-down in the protection level will occur. If  $\alpha > 0$  there will be an upward spiral in the protection level.

In all other cases we are not entirely sure how  $L^k$  behaves.

We will now give two specific examples in which this affine updates method is used. In the first case we assume that the true distribution of demand is normal with a known variance and our goal is to estimate the mean using the observed data. In the second case we assume that the true distribution is exponential and also here, we are aiming to estimate the mean.

Cooper et al (2006) consider the fact that that using a normal distribution for the demand is not very realistic. However, after mentioning a number of articles in which the normal distribution has been used, they conclude the following:

*“Thus, although a model with normally distributed observed quantity is unrealistic, such models have been used many times, and therefore we think that it is of interest to take a closer look at the dynamic behavior of such a model.”*

For both distributions that we will now examine, we assume that the mean of the observed quantity  $X^k$  (conditional on  $\mathcal{F}^{k-1}$ ), is equal to  $L^k$ . We will now derive the parameters  $\alpha$  and  $\beta$  for the two variations of the affine updates method. Based upon that we can give the exact conditions for the spiral-down effect.

### 2.3.1.1 Normal distribution

Following cooper et al (2006), in this section,  $N(\mu, \sigma^2)$  denotes the normal distribution with mean  $\mu$  and variance  $\sigma^2$  and also its cumulative distribution function. Thus, if  $A$  is a distribution then  $A \sim N(\mu, \sigma^2)$  means that  $A$  is normally distribution with mean  $\mu$  and variance  $\sigma^2$ . Moreover if  $B$  is a

distribution function, then  $B = N(\mu, \sigma^2)$  means that  $B$  is equal to the normal cumulative distribution function.

Let  $X$  be a sequence of the observed class-1 demand values and  $L$  be a sequence with the chosen protection levels. Given that the sequence  $L$  has started with a protection level  $L^0$ , we assume that:

$$X^k \sim N(L^{k-1}, \sigma^2).$$

We also assume that the revenue manager knows this and knows the exact value of  $\sigma^2$ . Then following our notation:

$$G(L^{k-1}, \cdot) = N(L^{k-1}, \sigma^2).$$

Let  $M^k$  denote the mean of the observed demand for the first  $k$  periods. Given the mean in period  $k - 1$  we can compute the mean in period  $k$  using the observed demand in this period:

$$M^k = \frac{k-1}{k} M^{k-1} + \frac{1}{k} X^k.$$

This  $M^k$  can then be used to construct an estimate of the distribution function of the demand:

$$\hat{H}^k = N(M^k, \sigma^2).$$

Then using the Littlewood rule a new protection level is determined using  $\hat{H}^k$ :

$$L^k = (\hat{H}^k)^{-1}(\gamma).$$

Now let  $\Phi$  denote the standard normal cumulative distribution function and  $\alpha = \sigma\Phi^{-1}(\gamma)$ , then:

$$L^k = M^k + \alpha.$$

We can also write the equation above as  $M^{k-1} = L^{k-1} - \alpha$  and then fill this in the equation for  $M^k$ :

$$M^k = \frac{k-1}{k} (-\alpha) + \frac{k-1}{k} L^{k-1} + \frac{1}{k} X^k.$$

Then substituting this into the equation for  $L^k$ :

$$L^k = \frac{k-1}{k} L^{k-1} + \frac{1}{k} X^k + \frac{1}{k} \alpha.$$

Since we assumed that  $\mathbb{E}(X^k | \mathcal{F}^{k-1}) = L^{k-1}$ , it follows that:

$$\mathbb{E}(L^k | \mathcal{F}^{k-1}) = L^{k-1} + \frac{1}{k} \alpha.$$

So if we compare this to the general inductive equation for the affine updates, we find that  $\beta = 1$ . We can now draw conclusions with respect to the way  $L^k$  behaves as  $k$  goes to infinity. First we need to establish that Assumption A holds, similar to Lemma 1 in Cooper et al (2006):

*If the system evolves according to the equations given above, then Assumption A holds.*

Now we can conclude the following, according to proposition 6 in Cooper et al (2006):

- (3) If  $\alpha < 0$ , then  $\gamma < 1/2$  and  $L^k \rightarrow -\infty$  with probability 1 as  $k \rightarrow \infty$ . This means there will be a downward spiral on the protection level.
- (4) If  $\alpha > 0$ , then  $\gamma > 1/2$  and  $L^k \rightarrow +\infty$  with probability 1 as  $k \rightarrow \infty$ . This means there will be an upward spiral on the protection level.
- (5) If  $\alpha = 0$ , then  $\gamma = 1/2$  then  $L^k \rightarrow Y$  with probability 1 as  $k \rightarrow \infty$  and  $Y$  being a random variable. In this case,  $Y \sim N(L^0, \sigma^2 \sum_{i=1}^{\infty} 1/i^2)$ .

In practice, it is often the case that  $f_2/f_1 > 1/2$ , which means that  $\gamma < 1/2$ . Therefore it is not unrealistic that a spiral-down effect could occur in those practical situations.

### 2.3.1.2 Exponential distribution

Again, let  $X$  be a sequence of the observed class-1 demand values and  $L$  be a sequence with the chosen protection levels. Given that the sequence  $L$  has started with a protection level  $L^0$ , we assume that:

$$X^k \sim \exp\left(1/L^{k-1}\right).$$

We assume that the revenue manager knows this and will therefore use the sample average of the  $X$  values to estimate the supposed mean of  $X$ . We can then derive the following equations in a similar fashion as we did for the normal distribution.

$$M^k = \frac{k-1}{k} M^{k-1} + \frac{1}{k} X^k,$$

$$\hat{H}^k = \exp\left(1/M^k\right).$$

The estimated distribution of class-1 demand ( $\hat{H}$ ) can now be used in the formula, defined by the Littlewood rule, to determine the protection level for period  $k$ :

$$L^k = (\hat{H}^k)^{-1}(\gamma) = \ln\left(\frac{1}{1-\gamma}\right) M^k.$$

Now rewrite the formula above to give an equation for  $M^{k-1}$ :

$$M^{k-1} = \frac{L^{k-1}}{\ln\left(\frac{1}{1-\gamma}\right)}.$$

This can then be filled into the equation for  $M^k$ :

$$M^k = \frac{k-1}{k} \frac{L^{k-1}}{\ln\left(\frac{1}{1-\gamma}\right)} + \frac{1}{k} X^k.$$

And then finally we can write the protection level formula in terms of the protection level of the previous period, using the assumption  $\mathbb{E}(X^k | \mathcal{F}^{k-1}) = L^{k-1}$ .

$$\mathbb{E}(L^k | \mathcal{F}^{k-1}) = \frac{k - 1 + \ln\left(\frac{1}{1 - \gamma}\right)}{k} L^{k-1}.$$

When comparing this to the general equation for the affine updates method, we can conclude that for the exponential case  $\beta = \ln(1/(1 - \gamma))$  and  $\alpha = 0$ . Also here, we need to establish that Assumption A holds, similar to Lemma 2 in Cooper et al (2006):

*If the system evolves according to the equations given above, then Assumption A holds.*

We can now reconstruct proposition 7 from Cooper et al (2006):

- (6) If  $\beta < 1$  then  $1 - \gamma = f_2/f_1 > 1/e$  and  $L^k \rightarrow 0$  with probability 1 as  $k \rightarrow \infty$ . This is the case in which a spiral-down occurs on the protection level.
- (7) If  $\beta = 1$  then  $1 - \gamma = f_2/f_1 = 1/e$  and  $L^k \rightarrow Y$  with probability 1 as  $k \rightarrow \infty$ , where  $Y$  is a finite random variable.
- (8)  $\beta > 1$  then  $1 - \gamma = f_2/f_1 < 1/e$  then  $f^k \rightarrow 0$  as  $k \rightarrow \infty$ . If  $\Omega$  is the sample space, this means that for those  $\omega \in \Omega$  such that  $f^k L^k(\omega) \rightarrow Y(\omega) > 0$  it holds that  $L^k(\omega) \rightarrow \infty$ . However if  $Y(\omega) = 0$ , then we cannot judge about the asymptotic behavior of  $L^k$ .

In practice it is realistic for  $f_2/f_1$  to be larger than  $1/e$ . Therefore, also for the exponential case, the spiral down effect could certainly occur in practical situations.

### 2.3.2 Empirical distribution

In this section we will study the behavior of the protection level when using the empirical distribution as a forecasting method. In section 2.2 we have already showed in an example how the spiral-down effect occurs when class-1 demand is assumed to be deterministic. Now we will study the case where class-1 demand is random and denoted by  $D$ . Recall that in the example in section 2.2 we had already derived that  $X := (D - (c - L)^+)^+$ , for a given protection level  $L$ .

Now consider the situation in which protection level  $L^{k-1}$  is used in period  $k$ , the random demand in that period is  $D^k$  and the observed class-1 demand is  $X^k$ . Given the observed values  $X^1 \dots X^k$  the revenue manager constructs the empirical distribution  $\hat{H}^k$ . The protection level  $L^k$  is then determined by computing  $\gamma$ -quantile from  $\hat{H}^k$ .

Let us look into the empirical distribution in more detail, to find out the exact value of  $L^k$  in terms of the observed values  $X^1 \dots X^k$ . Let  $X^{1:k} \dots X^{k:k}$  be the order statistics of  $X^1 \dots X^k$ . We can then state that  $L^k := X^{[k\gamma]:k}$ , which is position  $k * \gamma$  rounded up in the order statistics of  $X^1 \dots X^k$ . Next, we will compare  $L^k := X^{[k\gamma]:k}$  with  $L^{k+1} := X^{[(k+1)\gamma]:k+1}$  in two separate cases.

1.  $[(k + 1)\gamma] = [k\gamma]$ . That is, the positions in which the protection levels are found are the same for period  $k$  and period  $k + 1$ . Given the protection level in period  $k$ , the protection level in period  $k + 1$  now only depends on the observed demand  $X^{k+1}$  in that period and on  $X^{[k\gamma]-1:k}$ .

- a. If and only if the observed demand in period  $k + 1$  is at least  $L^k$ , then  $X^{[k\gamma]:k}$  is equal to  $X^{[(k+1)\gamma]:k+1}$ . This is illustrated in Figure 2.3.

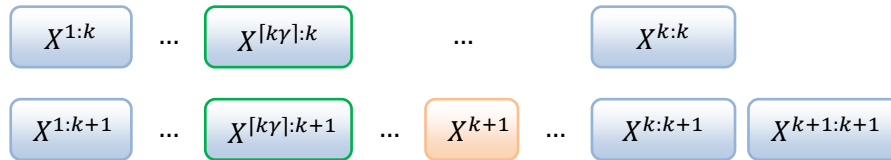


Figure 2.3: The order statistics before (top) and after (bottom) inserting the newly observed value for case (1.a).

This implies that:

$$X^{k+1} \geq L^k \leftrightarrow L^{k+1} = L^k.$$

- b. If and only if the observed demand in period  $k + 1$  is at least  $X^{[k\gamma]-1:k}$  and at most  $L^k$ , then the value position  $X^{[(k+1)\gamma]:k+1}$  is the observed demand itself. This case is illustrated in Figure 2.4.

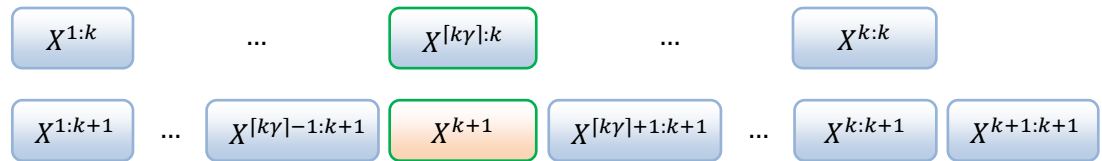


Figure 2.4: The order statistics before (top) and after (bottom) inserting the newly observed value for case (1.b).

Thus:

$$X^{[k\gamma]-1:k} \leq X^{k+1} \leq L^k \leftrightarrow L^{k+1} = X^{k+1}.$$

- c. If and only if the observed demand in period  $k + 1$  is at most  $X^{[k\gamma]-1:k}$ , then  $X^{[(k+1)\gamma]:k+1}$  is equal to  $X^{[k\gamma]-1:k}$ . This case is illustrated in Figure 2.5.

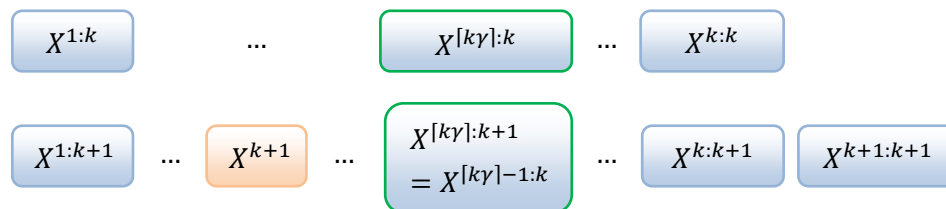


Figure 2.5: The order statistics before (top) and after (bottom) inserting the newly observed value for case (1.c).

Thus:

$$X^{k+1} \leq X^{[k\gamma]-1:k} \leftrightarrow L^{k+1} = X^{[k\gamma]-1:k}.$$

2.  $[(k + 1)\gamma] = [k\gamma] + 1$ . In this case the position in which the protection level is found in period  $k + 1$  is one higher than that in period  $k$ . Here, the protection level in period  $k + 1$  only depends on the newly observed value in that period and on  $X^{[k\gamma]+1:k}$  if we know what the protection level was in period  $k$ .
- a. If and only if the observed demand in period  $k + 1$  is at most  $L^k$ , then  $X^{[(k+1)\gamma]:k+1}$  is equal to  $X^{[k\gamma]:k}$ . This case is illustrated in Figure 2.6.

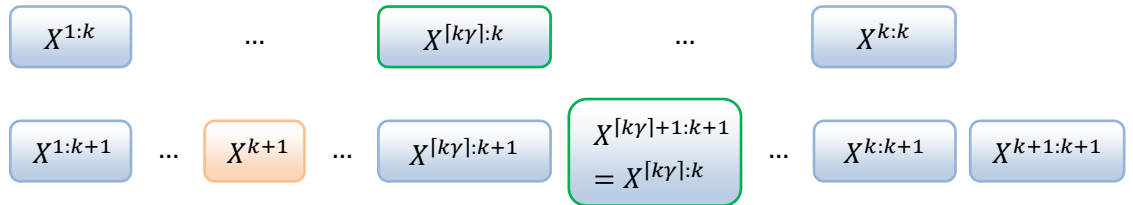


Figure 2.6: The order statistics before (top) and after (bottom) inserting the newly observed value for case (2.a).

Therefore:

$$X^{k+1} \leq L^k \leftrightarrow L^{k+1} = L^k.$$

- b. If and only if the observed demand in period  $k + 1$  is at least  $L^k$  and at most  $X^{[k\gamma]+1:k}$ , then  $X^{[(k+1)\gamma]:k+1}$  is equal to the observed demand itself. This case is illustrated in Figure 2.7.

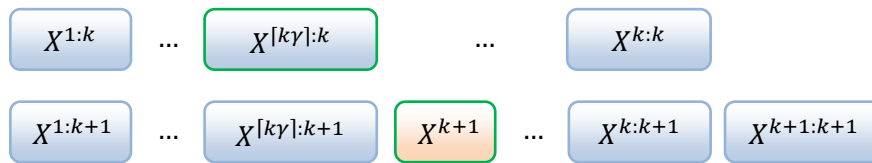


Figure 2.7: The order statistics before (top) and after (bottom) inserting the newly observed value for case (2.b).

Thus:

$$L^k \leq X^{k+1} \leq X^{[k\gamma]+1:k} \leftrightarrow L^{k+1} = X^{k+1}.$$

- c. If and only if the observed demand in period  $k + 1$  is at least  $X^{[k\gamma]+1:k}$ , then  $X^{[(k+1)\gamma]:k+1}$  is equal to  $X^{[k\gamma]+1:k}$ . This case is illustrated in Figure 2.8.

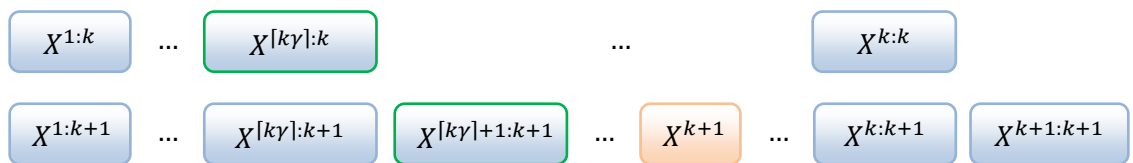


Figure 2.8: The order statistics before (top) and after (bottom) inserting the newly observed value for case (2.c).

Thus:

$$X^{k+1} \geq X^{\lfloor k\gamma \rfloor + 1:k} \leftrightarrow L^{k+1} = X^{\lfloor k\gamma \rfloor + 1:k}.$$

Since we know that  $X^{k+1} := (D^{k+1} - (c - L^k)^+)^+$ , we can compute the probabilities of the events on the left-hand side of the above stated cases. For any  $a \geq 0$  we know that:

$$\mathbb{P}(X^{k+1} > a | \mathcal{F}^k) = \mathbb{P}\left(\left(D^{k+1} - (c - L^k)^+\right)^+ > a \mid \mathcal{F}^k\right) = \mathbb{P}\left(D^{k+1} > a + (c - L^k)^+ \mid \mathcal{F}^k\right)$$

Now take  $a = L^k$ :

$$\mathbb{P}(X^{k+1} > L^k | \mathcal{F}^k) = \mathbb{P}\left(D^{k+1} > L^k + (c - L^k)^+ \mid \mathcal{F}^k\right) = \mathbb{P}\left(D^{k+1} > \max\{c, L^k\} \mid \mathcal{F}^k\right)$$

Using this we can explain the behavior of the protection level as  $k$  gets large for some particular cases. For example, if  $D^k < c$  with probability 1 for all  $k$ , then  $\mathbb{P}(D^{k+1} > \max\{c, L^k\} | \mathcal{F}^k) = 0$  and therefore  $\mathbb{P}(X^{k+1} \leq L^k | \mathcal{F}^k) = 1$ . If we look back at the cases we derived when comparing  $L^k$  to  $L^{k+1}$ , we can see that if  $X^{k+1} \leq L^k$  then  $L^{k+1} \leq L^k$ . This clearly signals the spiral-down effect. Note that the protection level will never be chosen below zero. This means that  $\{L^k\}$  is a bounded monotone sequence and thus there exists a random variable  $Y$  such that  $L^k \rightarrow Y$  as  $k \rightarrow \infty$ .

## 2.4 The spiral-down in practice: EMSRb

Belobaba (1987) developed a heuristic decision rule for determining protection levels for the single-leg problem with more than two classes. His method was called “Expected Marginal Seat Revenue” (EMSR). Later he came up with a model he called EMSRb, which is a slight modification of his own EMSR. The EMSRb model is smarter and handles multiple classes by comparing the revenue of the lower segment to a demand weighted average of the revenues of the higher segments.

Since EMSRb is widely used in the airline industry, it would be interesting to study it. We will first give a description of the model. Later this model will be used in some simulations to illustrate the spiral-down effect.

### 2.4.1 Model description

Let  $D_i$  be the random variable for the exogenous demand of class  $i$ , with  $i \in \mathbb{N}$ .  $H_i$  denotes the cumulative probability distribution function of  $D_i$ . Let  $f_i$  be the fare for class  $i$ . We denote the joint protection level for class  $i$  and higher with  $l_i$ . The marginal revenue is defined as the amount of extra revenue that is earned by selling one additional item. Now the expected marginal revenue that is gained when making the  $S_i$ th seat available for class  $i$  is:  $\text{EMSR}_i(S_i) = f_i * \bar{H}_i(S_i)$ , where  $\bar{H}_i(S_i) = 1 - H_i(S_i)$ . The EMSRb formulation to find the joint protection level  $l_i$  for class  $i$  and higher is as follows:

- Class-1:  
Find the value of  $l_1$  that makes  $\text{EMSR}_1(l_1) = f_1 * \bar{H}_1(l_1)$  equal to  $f_2$ . This means that  $l_1 = H_1^{-1}(1 - f_2/f_1)$ , which is the same as the Littlewood-rule we have been using throughout the first part of this chapter.



- Class-2:

Now we need to determine how many seats we want to jointly protect for classes 1 and 2 from class 3. In order to combine the higher classes into a joint protection, the following calculations need to be done. Let  $\bar{D}_i$  be the estimated mean of the random variable  $D_i$ . Then the combined average of class 1 and class 2 is defined as follows:

$$\bar{D}_{1,2} := \bar{D}_1 + \bar{D}_2.$$

We denote the estimated standard deviation for  $D_i$  by  $\hat{\sigma}_i$ . Now the combined standard deviation is defined as:

$$\hat{\sigma}_{1,2} := \sqrt{\hat{\sigma}_1^2 + \hat{\sigma}_2^2}.$$

Accordingly, the combined fare rate for classes 1 and 2 together can be defined as:

$$f_{1,2} := \frac{f_1 \bar{D}_1 + f_2 \bar{D}_2}{\bar{D}_{1,2}},$$

which is the demand weighted average of the fares. It makes sense to use this formulation since it reflect the expected revenue for classes 1 and 2 combined.

The complement of the combined cumulative distribution function is defined according to:

$$\bar{H}_{1,2}(d) := \text{Prob}[D_1 + D_2 \geq d].$$

Now we can define the combined expected marginal revenue:

$$\text{EMSR}_{1,2}(S) := f_{1,2} * \bar{H}_{1,2}(S).$$

The value for  $l_2$  is the one that makes  $\text{EMSR}_{1,2}(l_2) = f_{1,2} * \bar{H}_{1,2}(l_2)$  equal to  $f_3$ . This rule is somewhat intuitive. Namely as long as the combined expected marginal revenue for classes 1 and 2 is still higher than the fare for class 3, we want sell to classes 1 and 2. As soon as this does not longer hold, we would rather want to reserve the rest of the seats for class 3.

- Class-n:

Further to combine class n and higher into a joint protection, the following definitions are required:

$$\begin{aligned} \bar{D}_{1,n} &:= \sum_{i=1}^n \bar{D}_i \\ \hat{\sigma}_{1,n} &:= \sqrt{\sum_{i=1}^n \hat{\sigma}_i^2} \\ f_{1,n} &:= \frac{\sum_{i=1}^n f_i \bar{D}_i}{\bar{D}_{1,n}} \\ \bar{H}_{1,n}(d) &:= \text{Prob} \left[ \sum_{i=1}^n D_i \geq d \right] \end{aligned}$$

We can now find the value  $l_n$  that makes  $\text{EMSR}_{1,n}(l_n) = f_{1,n} * \bar{H}_{1,n}(l_n)$  equal to  $f_{n+1}$ .

## 2.4.2 Simulation

In order to illustrate and verify the mathematical results derived in the first part of this chapter, a program was build to simulate the effect. In this chapter we will use the widely applied model EMSRb to compute protection levels.

First we will describe the scenario that was simulated. Then we will briefly discuss the outcome of the simulation.

### 2.4.2.1 Scenario

The simulations for this thesis were performed in a similar way as in the paper by Cooper et al (2006). However, in their example they used a ‘stochastic approximation’ method for forecasting demand. In this paper we will use the ‘affine updates’ and the ‘empirical distribution’ forecasting methods. The affine updates method is used in combination with the normal distribution as well as the exponential distribution.

In our set-up, services are sold in two price-classes, class 1 and class 2. The assumption is that the class 1 is more expensive than class 2, so  $f_1 > f_2$ . These services are bought by three types of customers: type a, type b and type ab. Customers of type a will only buy class 1, type a customers buy class 2 and type ab customers buy the cheapest one that is available at the moment they arrive. The overall amount of services available for sale is fixed and defined by  $c$ .

The model used to compute protection levels is EMSRb. However, since there are only two booking classes, this gives the same results as applying the Littlewood-rule. This means that the conditions under which the spiral-down effect occurs derived in the section 2.3, still apply.

During a period of  $T$  time-units, customers of type  $i$  arrive according to a non-homogeneous Poisson process with rate function  $\lambda_i(t)$ , with  $t \in [0, T]$ . At the end of the period, the service is perished.

In order to reconstruct the spiral-down effect,  $K$  of these periods were simulated. At the start of every period  $k \in \{0, \dots, K - 1\}$  a protection level  $l_k$  is computed for protecting class-1 demand. The initial protection level  $l_0$  is given.

The simulation of the Poisson process is performed using discrete event simulation and works as follows. The time period from 0 to  $T$  is split into intervals of length 1. For every customer type, the number of arrivals per interval  $s$  is determined by a random sample from the Poisson distribution with parameter  $\mu_i(s)$ , where:

$$\mu_i(s) = \frac{1}{2}(\lambda_i(s) + \lambda_i(s + 1)) \quad \text{for } s \in \{0, \dots, T - 1\}, i \in \{a, b, ab\}$$

Since we will be using linear rate functions,  $\mu_i(s)$  is the average arrival rate during the interval.

Knowing how many arrivals will take place in each interval, we can determine at what point in the interval these events will occur. This will be done by taking random samples from the Uniform(0,1) distribution. The resulting overall arrival time of a customer arriving in interval  $s$  is then equal to  $s + U$ , where  $U$  is the random sample from the Uniform(0,1) distribution.

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We can combine the arrival times for all customer types in a single list. By putting this list in increasing order by arrival time, a chronological timeline is created containing every event throughout the time-period. Now all there is to do is work through this list from top to bottom, and simulating the customer behavior for each arrival. This means that we keep track of the number of services that are sold and based upon that, a customer will either buy in class 1, class 2 or not at all. This of course depends on current availability of the service in each class.

In the simulations we used time periods of length  $T = 100$ . The total availability of services that can be sold in each period was  $c = 100$ . The arrival rate functions were chosen as follows:

$$\lambda_a(t) = 0.005 * t,$$

$$\lambda_b(t) = 0.5 - 0.005 * t,$$

$$\lambda_{ab}(t) = 0.5,$$

for  $t = 1...100$ .

This means that the arrival rate of type-a customers increases over time, which makes sense. For example in the airline industry first class passengers tend to buy their tickets later for matters of flexibility. The fact that the arrival rate for type-b customers decreases over time is also intuitive. Take leisure passengers in airplanes for example. They usually book their tickets more time in advanced, so that they can get them for the lowest fare. The expected number of type-ab customers could be seen as a combination of both a and b.

When looking at the arrival rate of all customers together, we can see that:

$$\lambda_a(t) + \lambda_b(t) + \lambda_{ab}(t) = 0.5 + 0.5 = 1$$

This means that on average there is 1 customer per unit of time arriving. Since  $T = 100$ , the total expected number of customers is 100.

The prices for classes 1 and 2 were chosen in two different ways. Remember from section 2.3, we showed how the prices can influence whether or not the spiral-down effect occurs. We found that:

$$f_2/f_1 > 0.5$$

for the affine updates method using the normal distribution and:

$$f_2/f_1 > 1/e$$

for the affine updates method in combination with the exponential distribution.

Therefore it would be interesting to simulate a case where these conditions do apply (1) and a case where they do not (2). The prices were set as follows:

$$(1) f_1 = 1 \text{ and } f_2 = 0.7$$

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(2) and

For the rest of this paper we will refer to these cases as case (1) and case (2).

Let the number of services sold in the classes 1 and 2 be denoted by respectively  $n_1$  and  $n_2$  and given the prices  $p_1$  and  $p_2$ , then the revenue is defined as follows:

We assume in this simulation that the initial protection level for class 1 produces the maximum revenue. In order to find this optimal protection level, the average revenue was computed for every possible value of the protection level between 0 and 100. The results from these computations are shown in Figure 2.9 and Figure 2.10 for respectively cases (1) and (2).

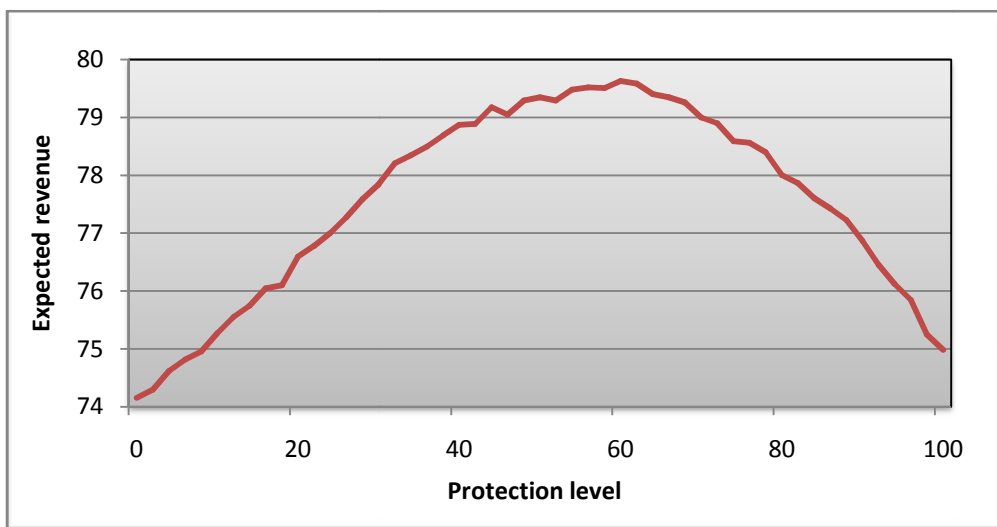


Figure 2.9: The mean revenue for case (1) per protection level, resulting from 1000 simulations

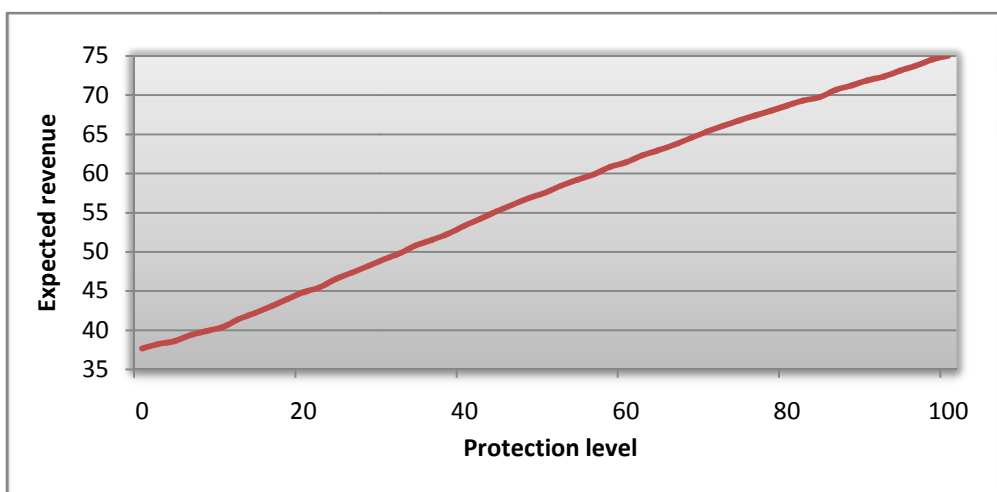


Figure 2.10: The mean revenue for case (2) per protection level, resulting from 1000 simulations

We can see in Figure 2.9 that first the revenue increases with the protection level and then goes down. The turning point seems to be around 58. For this reason, we have chosen the initial protection level  $L_0$  to be 58.

The same reasoning can be done for case (2) from Figure 2.10. Here we see that the expected revenue is only increasing in the protection level. The reason for this is that the price difference between the two fare classes is relatively big in comparison to case (1). This way, the missed revenue of turning away a class-1 customer is so high that class-2 customers should not be served at all. Therefore we chose the maximum value for  $L_0$ , which is 100.

The ‘affine updates’ method in combination with a normal distribution requires an extra parameter. This is the variance of the class 1 demand which is assumed to be normal. In chapter 2 we suggested that the revenue manager knows the exact value for the variance. However, in practice giving a good estimation for the variance is very hard, because it requires a lot of demand data. In our case, it is easy to generate a large set of demand data, since we can do as many simulations as we require. Therefore we ran 1000 single period simulations with given protection levels and collected the class 1 demand. The variance we computed over these numbers was 75 and used this as the variance-parameter for the assumed normal distributed class-1 demand. It is important to note though that the choice of this parameter influences the outcome of the simulations. The spiral-down could be much stronger or weaker for different values.

For the simulated cases we assumed that every arriving customer is observed. In practice however, this is not always the case. When for example the number of seats that are sold has reached the capacity of the airplane, demand for that flight is no longer observed.

The simulations were done 500 times. The results presented in this thesis are the averages taken from those simulations. Doing so, the graphs and numbers shown have a higher significance.

#### **2.4.2.2 Results**

With the configuration described in the previous section, simulations were performed. The results for case (1) are summarized in Figure 2.11.

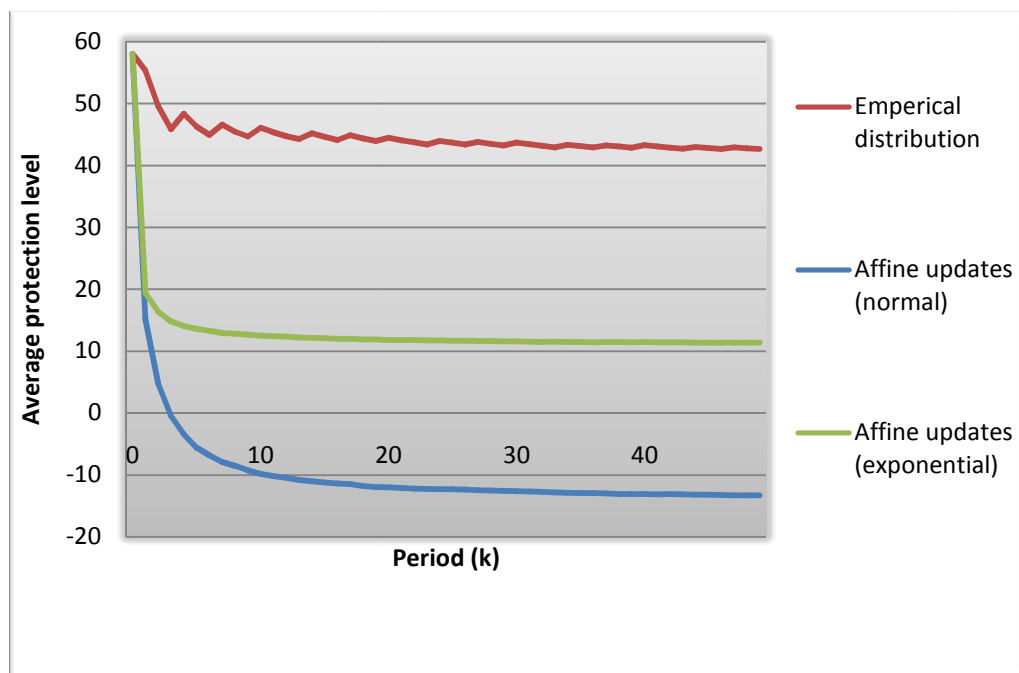


Figure 2.11: The spiral-down effect under three different forecasting methods

In this figure, we can see the simulated booking period on the horizontal axis and the average protection level resulting from 500 simulations on the vertical axis. As we can see, there clearly is a spiral-down occurring for every forecasting method. The effect seems to be the worst for the normal version of the affine updates. It starts off with the optimal protection level of 58 and goes all the way down to a negative protection level of -12. Using a negative protection level has the same effect as using a protection level equal to 0. When referring back to Figure 2.9, the expected revenue for such a protection level is only 74. This results in a 7 % revenue loss which is already reached after less than 5 periods.

Remarkable is that the largest decrease in the protection level happens in the first few periods. This can be explained by the way the forecasting methods work. Since they do not have any knowledge about the history of the demand before period 0, they are very vulnerable to the statistics that are achieved during the first periods. As there is a lot more history known at larger values of  $k$ , the forecasting methods are then less sensitive to newly observed demand data.

In Cooper et al (2006) a similar result was presented, however in their simulation they used a 'stochastic approximation' method to forecast demand. When comparing their results to the ones presented here, we notice that the spiral-down evolves much slower in their simulation. Their results show a pretty linear decrease over the first 800 periods. Then the decrease slows down and seems to converge after approximately 2000 periods. It is hard to tell where exactly the difference in results comes from, since we have not studied the stochastic approximation method for this thesis. However, obviously this method is less sensitive to the observed demand.

The results for case (2) are shown in Figure 2.12. According to the derived conditions, there should not be a downward spiral occurring for both of the affine updates techniques. When we look at the graph, we can conclude that this is indeed the case. The protection levels under both methods seem

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to spiral-up rather than down. For the empirical distribution method there is still a spiral-down. In section 2.3.2 we were not sure in how much the occurrence of the spiral-down effect is influenced by the prices for this method. It is therefore interesting to see that with a change in prices the problem is still occurring.

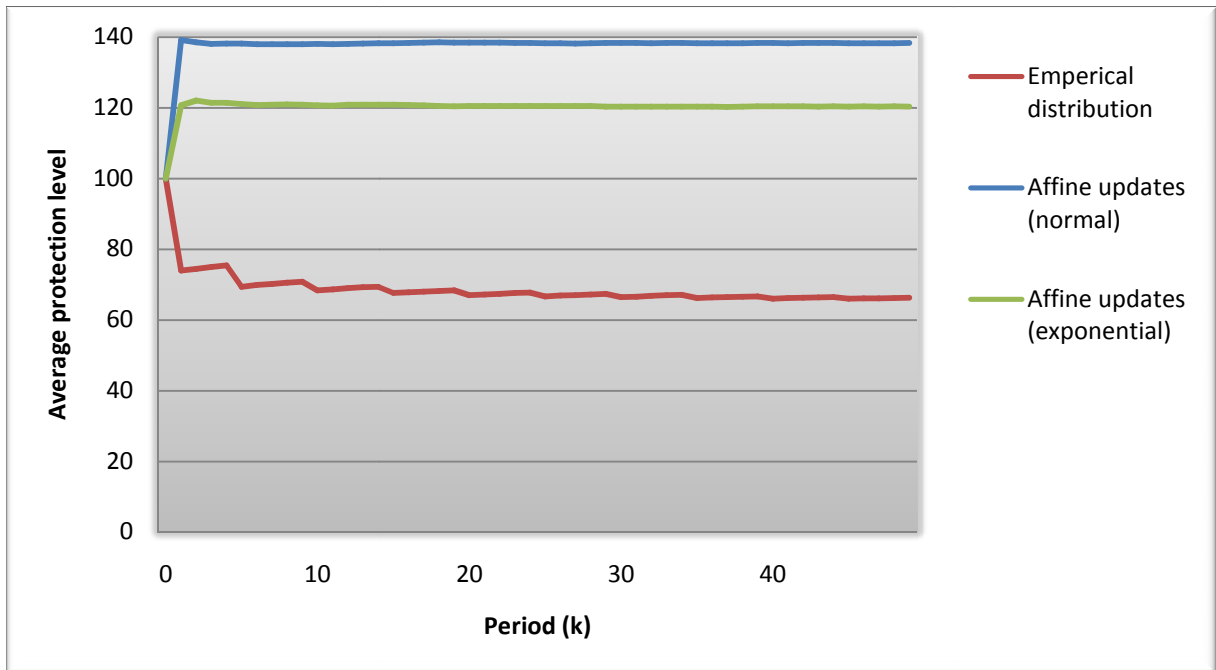


Figure 2.12: Simulation results case(2)

### 3. Preventing the spiral-down effect

The consequences of the spiral-down effect in practical situations could be problematic. From the simulations in section 2.4.2 we saw significant decreases in revenue over time. Therefore it would be of great value to study ways in which we could prevent this phenomenon from happening. The problem is however, that at this moment there is not much published work that is aimed towards solving this problem. Cooper et al (2006) does point out some possible solutions, but never actually analyses them.

Recall that one of the main causes is the fact that the model used to compute protection levels is based on erroneous assumptions. Therefore changing the model into one that is more suitable to its context could lead to better results. Cooper et al (2006) suggested two alternative models that might do the trick. The first one is a modification of EMSRb. This so-called ‘diversion’ model incorporates the possibility that lower fare-class customers buy in a higher fare class. It makes sure that fewer customers are allowed in lower fare classes given the probability that they are willing to upgrade to a higher fare-class. The second model is the general discrete choice model, developed by Talluri & Ryzin (2004). This model does not require the assumption of exogenous demand. For this thesis we have only implemented and simulated the modified EMSRb incorporating customer diversion. It would be interesting however, to study how the choice model performs under the same circumstances.

#### 3.1 EMSRb with diversion

The EMSRb model works under the assumption that the demand in each fare class is exogenous. However, this is not always the case. Consider the situation where not all customers are interested in only one fare-class. Some of them will try to buy in a lower fare class, but when there is no more availability they are willing to buy in a higher fare class. This means that the demand in the higher fare-classes depends on the demand in the lower ones. Thus the assumption of demand in each fare-class being exogenous no longer holds.

The phenomenon of customers buying in a higher fare class than they were originally interested in, is what Belobaba & Weatherford (1996) called ‘diversion’. In their paper they present an extended version of the EMSRb model that takes diversion into account. Based on the probability that a customer upgrades from one class to the other, it allocates less availability to lower price classes. Above we saw that diversion causes the assumption of exogenous demand to be incorrect. The model by Belobaba & Weatherford (1996) still requires this assumption, but seems to adjust the protection levels taking into account that not all customers are interested in a single fare class. Therefore, their model might prevent the spiral-down effect.

We will first give the theoretical description of the model as it is defined by Belobaba & Weatherford (1996). Then we will use it while simulating the same scenario as we did in chapter 2.

##### 3.1.1 Model description

In this section we give the result of the derivations done in Belobaba & Weatherford (1996). In their paper, they use a decision rule that incorporates diversion for extending the EMSRb model. Recall



from chapter 2 that  $f_i$  is the fare for class  $i$ ,  $\bar{D}_i$  is the mean of the exogenous random demand for class  $i$  and. Furthermore:

$$\bar{D}_{1,n} := \sum_{i=1}^n \bar{D}_i,$$

the joint mean demand for class 1 through class  $n$ . Also:

$$f_{1,n} = \frac{\sum_{i=1}^n f_i \bar{D}_i}{D_{1,n}},$$

which is the weighted average of the fares for class 1 through  $n$ .

For this model we introduce an extra parameter  $SU_{n+1,n}$ . This parameter is defined as the percentage of all customers who are able to buy in class  $n$ , but will buy up to class  $n+1$  if no more tickets are available in class  $n$ . There does not exist a closed form expression for  $SU_{n+1,n}$ . When this model is used in practice this parameter would have to be estimated. Belobaba & Weatherford (1996) mention that this is sometimes done by experiments of randomly closing down a lower class and observing the choice the customer makes.

Now the protection level for class  $n$   $L_n$  has to be chosen such that:

$$\bar{H}_n(L_n) = \frac{f_{n+1} - f_{1,n} SU_{n+1,n}}{f_{1,n}(1 - SU_{n+1,n})}$$

Where  $H_n$  is the estimated probability distribution of the demand for class  $n$  and  $\bar{H} = 1 - H$ .

### 3.1.2 Simulation

In this section we will analyze the extended EMSRb model to see if it prevents a spiral-down on the protection level. Just as in the previous chapter we will do this by performing simulations. In section 2.4 we knew exactly for each forecasting method under what conditions the spiral-down effect would occur. However, since these conditions were derived using the fact that the Littlewood rule was used, they may no longer apply.

#### 3.1.2.1 Scenario

In order to make a comparison with the results in chapter 2, most of the parameters for the simulations performed here will be kept the same.

In chapter 2 we simulated two cases in which case (1) should cause a spiral down and case (2) should not. This was based on the conditions we derived for the Littlewood rule in section 2.3. In this chapter we are looking for a method that prevents the spiral-down effect. Therefore we will only apply the extended EMSRb on case (1). Since case (2) does not always cause a spiral-down, it is not of the interest of this chapter to simulate its results under the extended EMSRb.

The admission policy given a certain protection level has not changed. The only difference is the way the protection level is determined. Therefore the expected revenue per protection level will be the

same as in chapter 2. Recall that in those simulations we used for case (1) and. This will remain unchanged for this scenario.

In the diversion model we have one extra parameter: . Recall that Belobaba & Weatherford (1996) defined this as the percentage of all customers who are able to buy in class n, but will buy up to class n+1 if no more tickets are available in class n. However it is doubtful what is meant by ‘being able to buy in class n’. In our two-class example we have type-a customers who are only interested in buying in class-1. The question is if these customers should also be seen as customers who are able to buy in class 2. They may be financially able to, but they are just not interested in buying class 2. For this reason, we have simulated the model for different values of .

As we can learn from section 2.4.2.1 the average arrival rate for type-ab customers is 0.5 and for type-a customers this is 0.25. The overall average arrival rate is equal to 1. This means that if type-a customers need to be taken into account, should be . If they do not need to be taken into account, should be . In the next section we will show results from simulations with values of chosen around those two numbers.

### 3.1.2.2 Results

In Figure 3.1 the simulation results can be viewed for case (1) using the EMSRb diversion model. It directly becomes clear that there is still spiral-down using the extension of EMSRb. Under all three of the forecasting techniques, the behavior of the protection level is similar to what we found in chapter 2. The affine updates method using a normal distribution gives even worse results in comparison to the standard EMSRb.

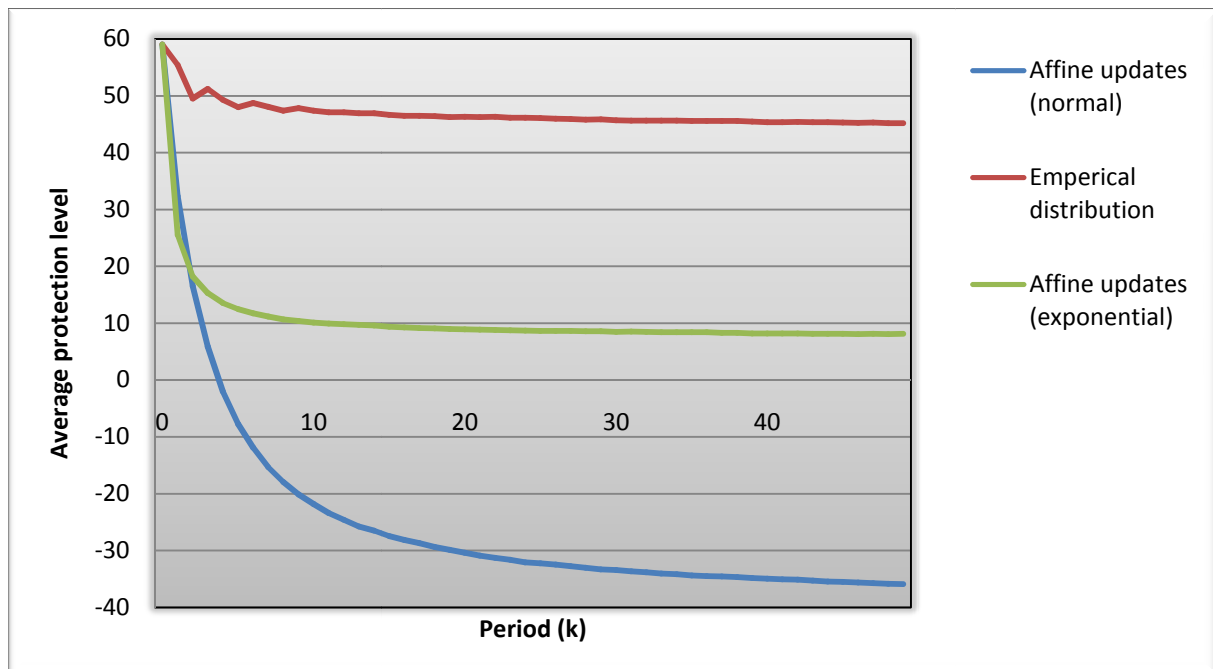


Figure 3.1: Results when simulating case (1) with the EMSRb diversion model using an upgrade percentage of 0.5.

The results in Figure 3.1 were generated using an upgrade percentage of 0.5. Since this is the main indicator for diversion in the model, it would be interesting to see what happens for higher values.

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Figure 3.2 and Figure 3.3 show simulation results generated using an upgrade percentage of respectively 0.6 and 0.7. It can be seen that as the upgrade percentage rises, the spiral-down effect seems to be less dramatic. For an upgrade percentage of 0.7, we even see the normal version of affine updates getting into an upward spiral and the empirical distribution method gives a pretty constant protection level over time. Due to the doubtfulness in the definition of this parameter, we cannot say for sure what value is the right one. Since the amount of time available for this part of the paper was very limited, we were not able to go into much more detail with this.

It is important realize that not all solutions preventing a spiral-down effect are preferable. Fixing the spiral-down effect is not the ultimate goal. Some models might rely less on observations made in the past and have a better dynamic behavior. However, such a model might be less accurate and therefore undesirable.

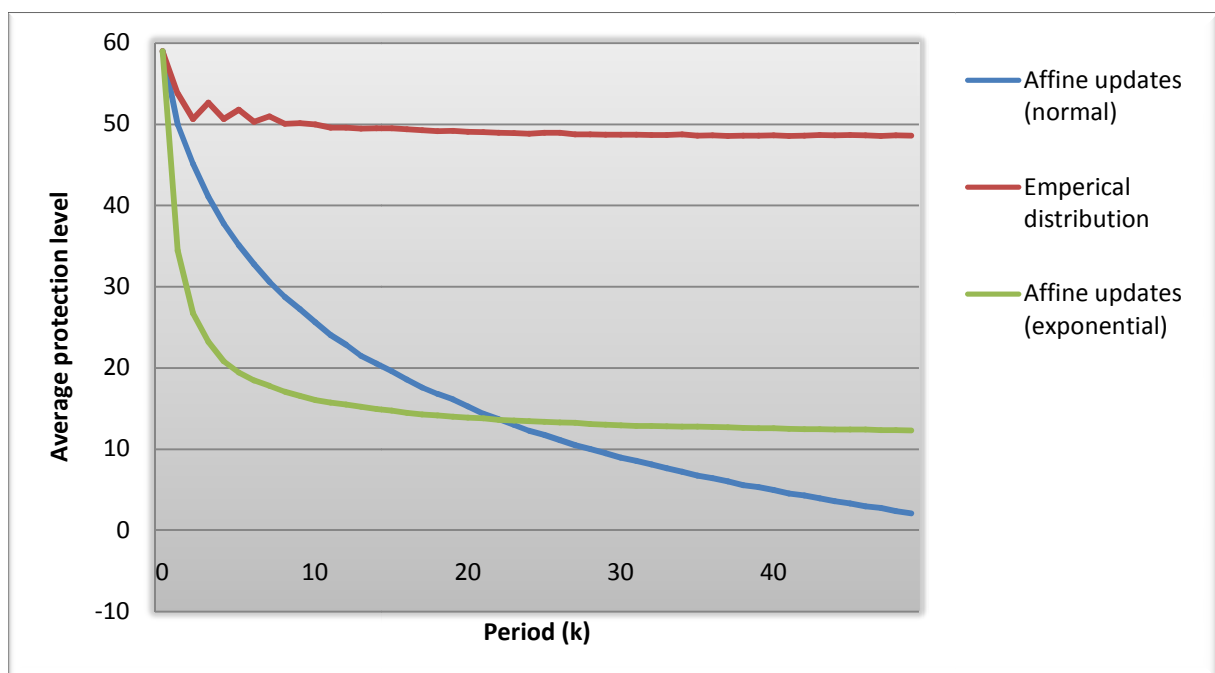
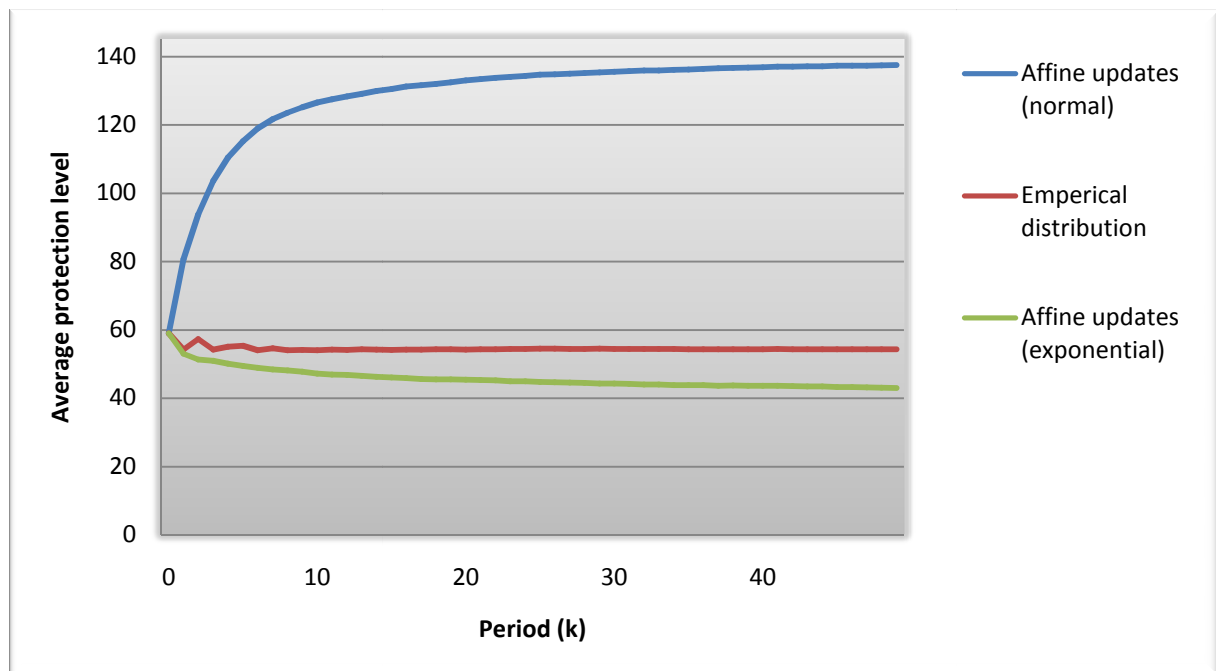


Figure 3.2: Simulation results for case (1) with an upgrade percentage of 0.6.

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*Figure 3.3: Simulation results of case (2) with an upgrade percentage of 0.7*

## 4 Conclusion

The goal of this thesis was to show what the spiral-down effect is, under what circumstances it occurs and to investigate a way of preventing it. This was accomplished by first explaining the process, in which the revenue manager continuously determines protection levels, observes and forecasts demand. Then we used a deterministic example to show how the spiral-down effect evolves. By analyzing this example, we saw that the problem has the following causes:

- The use of an incorrect model for determining the protection levels.
- Consequently using and updating the model.

Then we studied some specific forecasting techniques and showed the explicit conditions under which the spiral-down effect occurs. To emphasize the importance of the problem, we applied the widely used EMSRb model in a number of simulations. The results of this practical situation confirmed our theoretical findings about the problem.

In order to prevent the problem, we studied and simulated an extension of the EMSRb model, which incorporates diversion. Although some of the results improved, this alternative does not completely solve the problem.

While this thesis puts the emphasis on the iterative process of applying and updating revenue management models, most other papers only focus on analyzing a single situation. In those papers, the writers usually present a number of assumptions. They suppose that those assumptions hold and based on that they derive means to determine an optimal revenue management policy. This paper shows that more attention should be paid to what happens if those assumptions do not hold and the models are repeatedly updated and used.

The paper by Cooper et al (2006) was the fundamental source of inspiration. The contribution of this thesis is mainly to elaborate on their findings and to combine them with other published research. Also we have followed one of their suggestions, which was to study if the diversion based EMSRb model would prevent the spiral-down effect.

### 4.1 Further research

We have seen that the extended diversion based EMSRb model does not completely get rid of the problem. Since the time span for this thesis was quite short, we could not test any other methods. It would however be worthwhile to look into more alternative models, especially the ones that do not require the assumption of exogenous demand. One of those methods is the ‘general discrete choice model’ by Talluri and Ryzin (2004).

Also, it would be good to look into other forecasting techniques. In this thesis we have derived the exact conditions under which the spiral down occurs when using the affine updates technique or the empirical distribution to estimate demand. However, there are many other methods that have been researched and have been used in practice.

The theoretical results and the simulations that were presented in this paper were achieved for situations in which there are only two fare-classes. In reality, if you look for example to the airline industry, the number of fare-classes is much bigger. To obtain results that have even more

application in practical situations, it would therefore be interesting to study situations with more than two fare-classes. However, a theoretical approach for such situations might be quite complex. A good solution to this problem is to use simulations to derive the results.

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