Dealing with Idiosyncratic Credit Risk in a Finitely Granulated Portfolio

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Abstract

The credit VaR model that underpins the Basel II Internal Ratings Based (IRB) approach assumes that the portfolio idiosyncratic risk is fully diversified away and thus the required economic capital depends only on the systematic factor. Nevertheless that is not true in most cases. The impact of remaining firm-specific risks due to exposure concentrations on the portfolio VaR may be estimated by the usage of the so called Granularity Adjustments (GA). The scope of this paper is to mathematically present the newest and most accurate GA's, while trying to preserve the same notation, and to summarize which one works best in each particular situation.

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I. Introduction

Given a loan portfolio, credit risk may arise in general from two sources – systematic and idiosyncratic. Systematic risk reflects the effect of unexpected changes in the financial markets and the macroeconomic conditions on the borrowers' performance. Borrowers usually differ in the degree of sensitivity to the systematic factor, but almost no companies are invulnerable by the economic conditions in which they operate. That is the reason why the systematic portfolio risk is unavoidable and almost not diversifiable. On the other hand, idiosyncratic risk mirrors the effect of borrower-specific risks on the borrowers' performance. Those risks are specific for every individual obligor. The more fine-grained the portfolio is, the more the idiosyncratic risk is diversified away (by fine-grained we mean that even the largest exposures in the portfolio represent only a small part of the total portfolio exposure). In the framework of the Vasicek model, that we consider from now on, it is assumed that bank portfolios are all perfectly fine-grained (infinitely granulated). That means that the idiosyncratic (non-systematic) risk has been totally diversified away, in other words the capital requirements depend only on the systematic risks. On the contrary, in the real world, the bank portfolios are often finitely granulated and do not satisfy the asymptotic assumption. Exposure (name) concentration in the portfolio in general may be the result of imperfect diversification of the idiosyncratic risk due to the small size of the portfolio or the large size of some individual exposures. In case of substantial name concentration the Internal Ratings Based (IRB) formula (mentioned in the next chapter) omits the contribution of the remaining idiosyncratic risk to the required economic capital. This contribution may be assessed by means of methodology called Granularity Adjustment (GA). In reality banks use a variety of ways to measure concentration risk – some of them use sophisticated models that account for interactions between different kinds of exposure, others use simpler ad-hoc approaches. Some banks use stress tests that include a concentration risk component. A research of BCBS (2006) shows that for the tested portfolios name concentration adds between 2% and 8% to the Value at Risk (VaR). In general the Vasicek model produces higher risk measures due to the fact that it does not fully account for diversification between different credit-types in the portfolio. On the other hand, in some situations one-factor models produce lower risk measures, due to the fact that name concentrations are not captured. The scope of this paper is to mathematically describe the most accurate and recent granularity adjustment methods, while simultaneously trying to keep the notation the same. At the end of the paper those methods are compared and the various situations in which each of them may be applied are summarized.

II. Basel II and the Vasicek one-factor Model

In 1988 is taken the first step towards introducing standard capital adequacy for banks based on risk measurement – the Basel I Capital Accord. In general Basel Accords are recommendations and regulations issued by the Basel Committee of Banking Supervision (BCBS) – institution consisting of senior representatives of central banks and supervisory authorities from thirteen countries[1]. As the financial world (and in particular financial markets and instruments) is flourishing and rapidly changing ever since this moment, few years later banks already needs improved regulations to manage their risks. Because of that reason in 2007 the Basel II Capital Accord is introduced. It aims to improve the preceding regulations by means of making capital requirements more risk-sensitive, separating operational from credit risk and aligning economic and regulatory capital more closely to reduce the level of regulatory arbitrage.

Parts of the Basel II Capital Accord are the credit risk regulations – estimating required capital that covers the risk of banks' credit portfolios. This is accomplished by means of the so called Internal Ratings Based (IRB) Approach. The model behind IRB is known as the Asymptotic Single Risk Factor (ASRF) model and is based on the Vasicek model, introduced for the first time in 1991 and extended by others like Finger (1999), Gordy (2003), etc. In general the Vasicek model is a one–factor model that assumes normal distribution of both the idiosyncratic and systematic risk factors of any credit portfolio.

1. The Vasicek Model

In this part we briefly present the Vasicek one-factor model in mathematical terms for a given credit portfolio of n loans with maturity T. We assume that loan i (thus obligor i)

[1] Belgium, Canada, France, Germany, Italy, Japan, Luxemburg, Netherlands, Spain, Sweden, Switzerland, UK and US

has a probability of default p_i. In general we can characterize any borrower by three parameters:

- EAD_i - Exposure at default of loan i - measure of potential exposure in currency, calculated till maturity.
- LGD_i - Loss given default of loan i - the percentage of EAD_i that will not be recovered in case of default of loan i.
- PD_i - Probability of default of loan i (denoted simply by p_i) – the likelihood that loan i will not be repaid until maturity.

We define the default indicator on loan i to be D_i and to follow a Bernoulli distribution:

$$D_i = \begin{cases} 1, \text{ with prob. } p_i \text{ (if } i\text{-th borrower defaults)} \\ 0, \text{ with prob. } (1-p_i) \text{ (if } i\text{-th borrower does not default)} \end{cases}$$

We also define the effective exposure w_i of loan i and the relative effective exposure c_i of loan i in the portfolio by:

$$w_i = EAD_i \times LGD_i \text{ and } c_i = \frac{EAD_i \times LGD_i}{\sum_{i=1}^{n} EAD_i}.$$

Thus the absolute loss due to default of loan i is calculated by $L_i = w_i D_i$. Then the absolute loss on the whole portfolio is given by the weighted average of all separate absolute losses: $L = \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} w_i D_i$. On the other hand the relative loss of loan i is defined by $L_i^{rel} = c_i D_i$ and the portfolio relative loss is then $L^{rel} = \sum_{i=1}^{n} L_i^{rel} = \sum_{i=1}^{n} c_i D_i$. In his work Vasicek assumes that the LGD of each loan is 100% (it is defined to be loss before recoveries) and because of this we get $\sum_{i=1}^{n} c_i = 1$.

***Wiener process**: A continuous-time stochastic process $W(t)$ for $t > 0$ with $W(0) = 0$ and such that the increment $W(t) - W(s)$ is Gaussian with mean 0 and variance $t - s$ for any $0 \le s < t$, and increments for non-overlapping time intervals are independent. Brownian motion (i.e., random walk with random step sizes) is the most common example of a Wiener process.*

***Geometric Brownian Motion (GBM):** A continuous-time stochastic process $G(t)$ the logarithm of which follows a Wiener process. $G(t)$ follows a GBM if: $\partial G(t) = \mu G(t)\partial t + \sigma G(t)\partial W(t)$ with μ - drift, σ - volatility and $W(t)$ - a Wiener process.*

We assume that obligor i defaults if his assets' value A_i falls below the amount B_i that he owes to the bank. Then the assets' value of company i may be modeled by a GBM:

$$\partial A_i = \mu_i A_i \partial t + \sigma_i A_i \partial W_i(t)$$

with $W_i(t)$ being a Wiener processes. Then the analytic solution of the above stated stochastic differential equation is: $A_i(t) = A_i(0) \times \exp(\mu_i t - 1/2\sigma_i^2 t + \sigma_i W_i(t))$.

Finally, the assets' value of borrower i at maturity T is given by:

$$A_i(T) = \exp\left[\log(A_i(0)) + \mu_i T - \frac{1}{2}\sigma_i^2 T + \sigma_i \sqrt{T} X_i\right] = A_i(0) \times \exp\left[\mu_i T - \frac{1}{2}\sigma_i^2 T + \sigma_i \sqrt{T} X_i\right].$$

The standardized asset log-returns X_i are standard normal variables and ρ_i is the correlation between the firm's assets and the common factor. Because of their joint normality, X_i can be decomposed to fit a one-factor model:

$$X_i = Y\sqrt{\rho_i} + Z_i\sqrt{1-\rho_i}\ .$$

The variables $Y, \{Z_i | i = 1,\ldots,n\}$ are standard normal and mutually independent. In this decomposition $Y\sqrt{\rho_i}$ stands for the i-th loan's systematic risk (company's exposure to the common factor) with Y being the portfolio common factor (for example economic index, some macroeconomic factor, etc.) for the time till maturity. The other part $Z_i\sqrt{1-\rho_i}$ of the decomposition stands for the firm-specific (idiosyncratic) risk.

The Vasicek model is widely used in the industries for risk management and capital allocation. The method calculates the distribution of credit portfolio losses and sets the required capital allocations to be equal to the 99,9% percentile of a limiting distribution.

The Assumption: Assume that all loans are of the same amount (homogeneous portfolio), have the same probability of default, and have a common asset correlation ($c_i = 1/n, PD_i = p, \rho_i = \rho, \forall i = 1,\dots n$). For a fixed common factor Y we may express the probability of portfolio loss conditional on Y. Thus for any loan we get the probability of default under the specified common factor:

$$p(Y) = P\left[D_i = 1 \middle| Y\right] = \Phi\left(\frac{\Phi^{-1}(p) - Y\sqrt{\rho}}{\sqrt{1-\rho}}\right),$$

where Φ stands for the cumulative normal distribution function. We may think of this in another manner - every common factor generates a unique economic scenario and under every scenario we can calculate the conditional probability of default and then weight each of them by its likelihood. In this way we get the unconditional default probability for each loan. In our case it is just the average of all conditional probabilities over all scenarios.

Under Y the portfolio losses L_i^{rel} are independent and identically distributed with finite variance. Then again under the common factor Y, the portfolio loss L^{rel} converges to its expectation $p(Y)$ as the number of loans in the portfolio goes up to infinity ($n \to \infty$). In other words:

$$\lim_{n\to\infty} P[L^{rel} \le x] = \lim_{n\to\infty} P[p(Y) \le x] = \lim_{n\to\infty} P[Y \ge p^{-1}(x)] =$$

$$= \lim_{n\to\infty} \Phi\left(-p^{-1}(x)\right) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}\right),$$

which shows the cumulative distribution function of losses on a very large portfolio.

Definition of VaR: According to Basel II Accord, Value at risk (VaR) is the measure to calculate capital requirements for a given confidence interval α. The value at risk corresponding to a confidence interval α is simply the α - quantile of the loss distribution (the distribution of L). In other words:

$$VaR_\alpha = \inf\{x | P(L \le x) \ge \alpha\}.$$

Definition of VaR Contribution: VaRC measures the contribution of each separate obligor to the total VaR of the portfolio. It is important, because it can apply constraints on larger credit exposures. Briefly speaking it is the conditional expectation of L_i, given that the portfolio loss is equal to the corresponding VaR:

$$VaRC_{i,\alpha} = E(L_i | L = VaR_\alpha) = w_i E(D_i | L = VaR_\alpha) = w_i \frac{\partial VaR_\alpha}{\partial w_i}(L).$$

It can be easily derived that the sum of all separate contributions is equal to the VaR of the portfolio, which is not true for the individual VaR's. That is the main reason for which VaRC's are considered:

$$\sum_{i=1}^{n} VaRC_{i,\alpha} = \sum_{i=1}^{n} w_i E[D_i \mid L = VaR_\alpha] = E\left[\sum_{i=1}^{n} w_i D_i \mid L = VaR_\alpha\right] = E[L \mid L = VaR_\alpha] = VaR_\alpha.$$

Thus Vasicek (1991) finally derives the VaR and VarC formulae under the Basel II Accord I in terms of the relative loss L^{rel} instead of the absolute loss L:

$$VaR_\alpha = \Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

$$VaRC_{i,\alpha} = \frac{1}{n}\Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

The transition to 'absolute' VaR may be done by multiplying the 'relative' one by the total portfolio exposure $n \times EAD$.

The general case: In the case of an inhomogeneous portfolio the above described convergence still holds (with one necessary and sufficient restriction). Then the portfolio loss L^{rel} under the condition Y converges to its expectation $p(Y)$ as before if the so called Herfindahl-Hirschman Index (explained in detail later) goes to zero as n goes to infinity:

$$HHI = \sum_{i=1}^{n} \left(\frac{EAD_i}{\sum_{j=1}^{n} EAD_j} \right)^2 \to 0 \text{ as } n \to \infty .$$

Thus we need to consider portfolios without any name concentrations (infinitely granulated portfolios), i.e. all the separate loans are of similar size and none of them is much larger than the others. For those portfolios we can assume that the limiting distribution given above is a good approximation of the portfolio loss distribution. In general it reproduces well the fat tail and skewness of the loss distribution.

Assume now that all loans have different probabilities of default PD_i as stated in the beginning of the chapter and different correlations ρ_i. In this case Huang et al. (2007) derive the general VaR and VarC formulae again, this time in terms of the absolute loss L:

$$VaR_\alpha = \sum_{i=1}^{n} w_i \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho_i}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}} \right)$$

$$VaRC_{i,\alpha} = w_i \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho_i}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}} \right)$$

III. Granularity Adjustment and Semi-Asymptotic Approaches

There are various methodologies proposed by practitioners and researchers that deal with name concentrations in credit portfolios. In general, those can be divided into two groups – methods based on heuristic measures of risk concentration and methods based on more sophisticated risk modeling. The second group is to be preferred if there exist feasible implementations of the models.

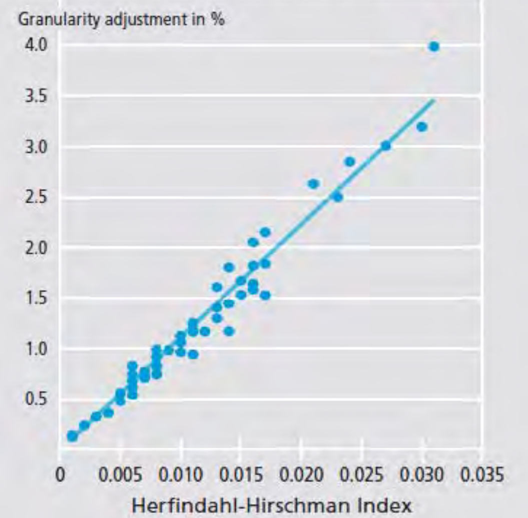
1. Ad Hoc Granularity Adjustment Approach

The simplest granularity adjustment is not based on any credit risk model. It uses the so called Herfindahl-Hirschman Index (HHI) of the loan portfolio:

$$HHI = \sum_{i=1}^{n} \left(\frac{EAD_i}{\sum_{j=1}^{n} EAD_j} \right)^2 ,$$

where EAD_i is as defined in the second chapter. For example for portfolio consisting of two loans of equal amounts the HHI is equal to $0{,}5^2 + 0{,}5^2 = 0{,}5$. It is considered that the closer the HHI to 1, the higher the concentration of the portfolio and so the higher the granularity adjustment. The shortfall is that there is no consistent rule that sets appropriate granularity add-on for a given HHI. In fact, empirical studies on single name concentration show that there can be established almost linear relation (see Figure 1) between the HHI and the percentage granularity adjustment (see Deutsche Bundesbank (2006)). Another shortfall of this approach is that it does not account for changes in the credit quality of the obligors (such changes have huge impact on the granularity adjustment on the required reserves).

Figure 1:
Link between the granularity adjustment* and the Herfindahl-Hirschman Index**



2. Vasicek Granularity Adjustment Approach

In 2002 Vasicek proposes a granularity adjustment approach that tries to increase systematic risk in order to compensate for the ignored (due to name concentrations) idiosyncratic risk. The GA assumes equal probabilities of default and correlations of the loans. The problem is that for small portfolios the law of large numbers does not hold and the convergence assumption of the Vasicek model also does not hold. In other words the conditional variance of the portfolio loss $Var(L^{rel}|Y)$ at maturity T is non zero. Let δ be the HHI of the portfolio. Then we can derive expressions for the conditional variance, unconditional expectation and unconditional variance of L^{rel} at time T (for details on the derivation refer to Vasicek (2002)). Having those and the limiting distribution underlying the Vasicek model, we can derive the percentage granularity adjustment to the distribution of L^{rel} at maturity:

$$P[L^{rel} \leq x] = \Phi\left(\frac{\sqrt{1-\rho-\delta(1-\rho)}\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho+\delta(1-\rho)}} \right).$$

This approach tries to increase the systematic risk and in this way to compensate for the idiosyncratic risk that is considered to be fully diversified, but is rather vague and inefficient, due to the fact that both risk types have different distributions. The model is known to perform poorly in practice and for this reason it is not mathematically explained in this paper and not compared to the newer and more sophisticated granularity adjustment approaches later on.

3. Emmer and Tasche Granularity Adjustment Approach

Gordy (2003) presented a methodology for estimating a capital add-on that is supposed to cover the undiversified idiosyncratic risk that remains in the credit portfolio. Emmer and Tasche (2003) improve further the approach by developing a new formula for computing adjustments at the transaction level. That formula calculates the partial derivatives of the estimated approximation of VaR_{α} and multiplies them with the portfolio weights. Thus we simply get approximated capital requirements for each separate asset in the portfolio, assuming again that every loan has LGD=100% – the so called VaR contributions (VaRC's)[2]:

[2] Notice that the VaRC formula stated by Emmer and Tasche is in terms of relative exposures like the one defined by Vasicek in the homogeneous case, unlike the VaRC's stated by the authors of the other GA approaches. The transition is done again by multiplying by the exposure of the particular loan.

$$\text{Capital Requirements for loan } i = c_i \frac{\partial VaR_\alpha}{\partial c_i}.$$

The method has several shortfalls:

- It assumes that the portfolio HHI, in this case equal to $\sum_{i=1}^{n} c_i^2$ goes to zero as $n \to \infty$. Thus its accuracy is not satisfactory when there are very large exposures in the portfolio. To deal with this problem we refer to the so called Semi-Asymptotic Approach, described later in the chapter.
- It does not maturity-adjust the input parameters.
- The formulae are too complex and hard to implement.
- It does not account for idiosyncratic recovery rates. The loan recovery rate is normally equal to one minus the loan LGD in percentage. The model assumes fixed LGD=100% and thus identical expected LGD's for each loan. To introduce recovery risk the LGD's should be drawn from some proper distribution like the gamma distribution. In general the model may be extended to cover this issue, but then the formulae get even more complex and hard to implement.

We skip the mathematical explanation of this model, because of the above stated shortfalls. We continue with the semi-asymptotic granularity adjustment approach, which is much simplified and easier to implement.

4. Semi-Asymptotic Granularity Adjustment Approach (SAGA)

The approach that we describe in this section is developed by Emmer and Tasche (2005) and is an alternative approach for measuring capital requirements for loans with higher concentrations (and thus is applied only on part of the portfolio), capturing fully the effect of those concentrations on the portfolio risk. Let us first introduce the new variables we need in order to mathematically present the model. We define the event ED_i (event of default) that loan i defaults, because its value falls below some critical threshold r_i:

$$ED_i = \left\{\sqrt{\rho_i}Y + \sqrt{1-\rho_i}Z_i \le r_i\right\}$$

Assume without any loss of generality that $\rho_1 = \tau$, $\rho_i = \rho$ for $\forall i = 2,\ldots,n$, $Z_1 = Z$, $r_1 = a$ and $r_i = r$ for $\forall i = 2,\ldots,n$. We also assume that the LGD of each loan is 100%, $c_1 = c$, $\sum_{i=2}^{n} c_i = 1$ and $\lim_{n\to\infty} HHI = \lim_{n\to\infty}\sum_{i=2}^{n} c_i^2 = 0$.

If we define $ED = \left\{\sqrt{\tau}Y + \sqrt{1-\tau}Z \le a\right\}$ to be the event of default of the first loan and $X = P\left[Z \le \frac{r-\sqrt{\rho}y}{\sqrt{1-\rho}}\right]\Big|_{y=Y}$, we can then give the following formulae for the portfolio loss (L) and the so called semi-asymptotic percentage loss function (PL):

$$L = c1_{\left\{\sqrt{\tau}Y+\sqrt{1-\tau}Z\le a\right\}} + (1-c)\sum_{i=2}^{n} c_i 1_{\left\{\sqrt{\rho}Y+\sqrt{1-\rho}Z_i\le r\right\}}$$

$$PL(c) = cD + (1-c)X$$

A proof of the latter equation is not the scope of this thesis and can be found in the appendix of Emmer and Tasche (2005). In general it gives us a model to apply separately for each loan (that may be considered a shortfall of the approach). The semi-asymptotic capital requirement (at level α) of the loan with exposure c (this may be any loan in the portfolio) is then defined as:

$$cP[ED \,|\, PL(c) = q_\alpha(PL(c))].$$

Having those separate asset charges we can add them up and get the capital requirements for the whole portfolio. The good thing about this method is that it accounts for the exposure c (in other words it is not portfolio invariant).

The last thing to do is to derive the solution for the given capital charges equation. Let us express the conditional probability of X given D by:

$$P\left[X \le x \,\middle|\, D = i\right] = F_i(x) = \int_{-\infty}^{x} f_i(t)dt,\ x \in \mathbb{R},\ i = 0,1,$$

given that F_i and f_i are the conditional distribution and density functions of X respectively and D is the default indicator of the first loan. Then we can easily derive the distribution function of the percentage loss:

$$P[PL(w) \leq z] = PD \times F_1\left(\frac{z-c}{1-c}\right) + (1-PD) \times F_o\left(\frac{z}{1-c}\right)$$

and its density function (given by the first derivative of the distribution function):

$$\frac{\partial P[PL(w) \leq z]}{\partial z} = \frac{1}{1-c} \times \left(PD \times f_1\left(\frac{z-c}{1-c}\right) + (1-PD) \times f_0\left(\frac{z}{1-c}\right)\right)$$

with $PD_1 = PD$ being the probability of default of the considered loan (in our case the first loan). One can use these formulae to numerically derive the α-quantile of the percentage loss ($q_\alpha(PL(c))$). Given our assumptions about the distributions of the factors Y and Z (remember, we assumed that they are both independent and standard normal) we can finally derive the formula for the percentage capital charges for the loan under consideration:

$$VaRC_\alpha = c \times P[ED | PL(c) = z] = c \times \frac{PD \times f_1\left(\frac{z-c}{1-c}\right)}{PD \times f_1\left(\frac{z-c}{1-c}\right) + (1-PD) \times f_0\left(\frac{z}{1-c}\right)}.$$

Because of the distribution assumptions we can now explicitly derive the conditional densities f_i and use them in the above expression:

$$f_1(z) = \begin{cases} \frac{1}{PD} \times f(z) \times \Phi\left(\frac{1}{\sqrt{1-\tau}} \times \left(\Phi^{-1}(PD) + \sqrt{\tau} \times \frac{\sqrt{1-\rho} \times \Phi^{-1}(z) - r}{\sqrt{\rho}}\right)\right), & z \in (0,1), \\ 0, \text{otherwise} \end{cases}$$

$$f_0(z) = \begin{cases} \frac{1}{1-PD} \times f(z) \times \Phi\left(-\frac{1}{\sqrt{1-\tau}} \times \left(\Phi^{-1}(PD) + \sqrt{\tau} \times \frac{\sqrt{1-\rho} \times \Phi^{-1}(z) - r}{\sqrt{\rho}}\right)\right), & z \in (0,1), \\ 0, \text{otherwise} \end{cases}$$

with

$$f(z)=\begin{cases}\sqrt{\frac{\rho}{1-\rho}}\times\exp\left(\frac{1}{2}\times\left(\left(\Phi^{-1}(z)\right)^2-\left(\frac{\sqrt{1-\rho}\times\Phi^{-1}(z)-r}{\sqrt{\rho}}\right)^2\right)\right), & z\in(0,1).\\ 0\text{ , otherwise}\end{cases}$$

However, there is one major shortfall of value at risk (VaR) as a portfolio risk measure. In the case $PD<1-\alpha$ we may concentrate all the exposure to the single loan with this low probability of default and this is going to reduce the capital charges. To avoid this, we may consider different and sounder measures like Expected Shortfall (ES) instead of VaR. More details about this limitation of VaR are given in Emmer and Tasche (2005), and Tasche and Theiler (2004).

5. Gordy and Lütkebohmert Granularity Adjustment (GA) Approach

In this part we introduce a granularity adjustment suitable to use under Pillar 2 of Basel II (Basel Committee on Bank Supervision 2006), developed by Gordy and Lütkebohmert in 2006. It is a CreditRisk^+ - based approach similar to the one published by Emmer & Tasche in 2005. The similarity comes from the fact that both models are based on the basic concepts and initial form of GA introduced by Gordy in 2000 for application in Basel II (published in 2003) and later refined by Wilde (2001b), Martin & Wilde (2003) and others. In principal the adjustment can be applied to any risk-factor model and is adapted to the latest changes in the definition of regulatory capital (finalized Basel II distinguishes between unexpected loss (UL) capital requirements and expected loss (EL) reserves). This means that the granularity adjustment for capital requirements should be invariant to EL and should not be expressed in terms of EL plus UL as in the past.

To estimate the required value at risk for the given portfolio we need $\alpha_q(L)$, that is the q-th percentile of the distribution of the portfolio loss. The IRB formula estimates the q-th percentile of the conditional expected loss - $\alpha_q(E[L\,|\,Y])$ and thus we define the exact granularity adjustment to be equal to the difference: $GA=\alpha_q(L)-\alpha_q(E[L\,|\,Y])$. This exact adjustment cannot be obtained, but can be approximated by a Taylor series in orders of

$1/n$. Let $\mu(Y) = E[L \mid Y]$ and $\sigma^2(Y) = Var[L \mid Y]$ are defined to be the conditional mean and variance of the portfolio loss, and h is the probability density function of Y. For $\varepsilon = 1$ we can express the portfolio loss as: $L = \mu(Y) + \varepsilon(L - \mu(Y))$. We use second–order Taylor expansion at $\varepsilon = 0$ to approximate the α - quantile of the loss as did Wilde (2001b):

$$\alpha_q(L) = \alpha_q(\mu(Y)) + \frac{\partial}{\partial \varepsilon}\alpha_q[\mu(Y) + \varepsilon(L - \mu(Y))]\Big|_{\varepsilon=0} + \frac{1}{2}\frac{\partial^2}{\partial \varepsilon^2}\alpha_q[\mu(Y) + \varepsilon(L - \mu(Y))]\Big|_{\varepsilon=0} + O(\varepsilon^3)$$

Gourieroux (2000) shows that the first derivative in the expansion vanishes and the second derivative is exactly the first-order granularity adjustment, given by the formula:

$$GA = \frac{-1}{2h(\alpha_q(Y))}\frac{\partial}{\partial y}\left(\frac{\sigma^2(y)h(y)}{\mu'(y)}\right)\Big|_{y=\alpha_q(Y)}.$$

We change notation slightly only for this chapter and set the weights w_i to be the proportion of the exposure of loan i to the total exposure of the portfolio: $w_i = \dfrac{EAD_i}{\sum_j EADj}$.

Then $L_i = LGD_i \times D_i$ and again $L = \sum_i w_i L_i$ become the relative losses of loan i and the whole portfolio respectively. We can express our granularity adjustment in terms of true *EL* and *UL*, and their asymptotic approximations EL^{as} and UL^{as}:

$$\alpha_q(L) - \alpha_q(E[L \mid Y]) = (UL + EL) - (UL^{as} + E[E[L \mid Y]]) = UL - UL^{as},$$

because $EL = E[L] = E[E[L \mid Y]]$.

The expressions $\mu(Y)$ and $\sigma^2(Y)$ depend on the model for the portfolio loss. It would be desirable to calculate the GA based on the Vasicek model, but then the conditional mean and variance are too complex (for supervisory application). This implies that we should use an indirect methodology – we initially base the calculations of the expressions on a different model and then change the input parameters in a specific way to restore the consistency of the overall adjustment as much as possible. The chosen initial model is the

CreditRisk^+. In general, in its standard version the recovery rates (and thus the LGD's) are assumed to be deterministic. However Gordy and Lütkebohmert allow in their model the LGD's to be random variables. According to the CreditRisk^+ model the conditional probabilities of default are given by:

$$P[D_i = 1 | Y] = PD_i(y) = PD_i(1 - l_i + l_i y),$$

where l_i is defined as a factor loading that controls the sensitivity of borrower i to the common factor Y, which we assume is gamma distributed with mean 1 and positive variance $1/\xi$. Also as part of the model we assume a Poisson distribution of the default indicator and that LGD_i is independent of Y for every loan. Thus we get:

$$\mu(y) = E[L | Y = y] = \sum_{i=1}^{n} w_i \mu_i(y) \text{ and } \sigma^2(y) = Var[L | Y = y] = \sum_{i=1}^{n} w_i \sigma_i^2(y), \text{ where:}$$

$$\mu_i(y) = E[L_i | Y = y] = ELGD_i \times PD_i(y) = ELGD_i \times PD_i(1 - l_i + l_i y),$$

$$\sigma_i^2(y) = Var[L_i | Y = y] = E[LGD_i^2 \times D_i | Y = y] - ELGD_i^2 \times PD_i^2(y) =$$

$$= E[LGD_i^2] \times E[D_i^2 | Y = y] - \mu_i^2(y).$$

In the given expressions $ELGD_i$ and $VLGD_i$ stand for the expected value and the standard deviation of LGD_i respectively. For the default indicator we have assumed a Poisson distribution so:

$$E[D_i | Y] = Var[D_i | Y] = PD_i(Y) \Rightarrow E[D_i^2 | Y] = PD_i(Y) + PD_i^2(Y).$$

We can also substitute $E[LGD_i^2] = Var[LGD_i] + E^2[LGD_i] = VLGD_i^2 + ELGD_i^2$ and get:

$$\sigma_i^2(y) = (VLGD_i^2 + ELGD_i^2) \times (PD_i(Y) + PD_i^2(Y)) - \mu_i^2(y) = C_i \mu_i(\alpha_q(Y)) + \mu_i^2(\alpha_q(Y)) \frac{VLGD_i^2}{ELGD_i^2}$$

where $C_i = \dfrac{ELGD_i^2 + VLGD_i^2}{ELGD_i}$.

By definition of the CreditRisk$^+$ model the EL required reserves for loan i are:

$$R_i = ELGD_i \times PD_i$$

and the UL required capital is given by:

$$K_i = E[L_i \mid Y = \alpha_q(Y)] = ELGD_i \times PD_i \times l_i \times (\alpha_q(Y) - 1)$$

In the CreditRisk$^+$ framework we have:

$$\mu_i(\alpha_q(Y)) = K_i + R_i \qquad \Rightarrow \qquad \mu_i'(\alpha_q(Y)) = \frac{K_i}{\alpha_q(Y) - 1}$$

$$\sigma_i^2(\alpha_q(Y)) = C_i \mu_i(\alpha_q(Y)) + \mu_i^2(\alpha_q(Y)) \frac{VLGD_i^2}{ELGD_i^2}.$$

Using those and given the granularity adjustment formula we can now substitute everything we have in it and get :

$$GA = \frac{-1}{2}\left(\frac{\sigma^2(\alpha_q(Y))h'(\alpha_q(Y))}{\mu'(\alpha_q(Y))h(\alpha_q(Y))} + \frac{\partial}{\partial y}\left(\frac{\sigma^2(y)}{\mu'(y)} \right) \Bigg|_{y=\alpha_q(Y)} \right),$$

with $\dfrac{\partial}{\partial y}\left(\dfrac{\sigma^2(y)}{\mu'(y)} \right)\Bigg|_{y=\alpha_q(Y)} = \dfrac{\sum_{i=1}^n w_i^2 \mu_i'(\alpha_q(Y))(C_i - 2\mu_i(\alpha_q(Y)))}{\sum_{i=1}^n w_i \, \mu_i'(\alpha_q(Y))}$.

Thus at the end we have the following expression for the granularity adjustment:

$$GA = \frac{1}{2K^*} \sum_{i=1}^n w_i^2 \left[\left(\delta C_i (K_i + R_i) + \delta (K_i + R_i)^2 \frac{VLGD_i^2}{ELGD_i^2} \right) - K_i \left(C_i + 2(K_i + R_i) \frac{VLGD_i^2}{ELGD_i^2} \right) \right],$$

where $K^* = \sum_{i=1}^{n} w_i K_i$ is the required capital per unit exposure for the whole credit portfolio

and where $\delta = (\alpha_q(Y) - 1)\left(\xi + \frac{1-\xi}{\alpha_q(Y)}\right)$ is a regulatory parameter, through which the variance parameter ξ of the systematic factor affects the GA. According to Basel II we set $q = 99{,}9\%$ and $\xi = 0{,}25$ (this value depends on the calibration method used) and thus we get $\delta = 4{,}83$.

To somewhat simplify the GA formula we assume that all second-order terms may be omitted, because the quantities they add to the GA are too small. Thus we get the simplified formula the accuracy of which is evaluated later:

$$\tilde{GA} = \frac{1}{2K^*}\sum_{i=1}^{n} w_i C_i (\delta(K_i + R_i) - K_i).$$

In the model discussed above there is only one implementation challenge – the aggregation of multiple exposures into a single one for each borrower. Gordy and Lütkebohmert (2007) propose a way to deal with this complication – the banks are allowed to calculate the GA based only on the set of largest exposures in the portfolio (m out of n). Still, the smaller exposures' influence should also be taken into account and this is accomplished by setting an upper bound of the GA for them. From the supervisory point of view that is not a very consistent method, because the upper bound in most cases is bigger than the "real" GA (and never smaller).

IV. Numerical Methods for Derivation of VaR in the Vasicek Model

In this chapter we describe the numerical methods for computation of value at risk (VaR) in the Vasicek one-factor portfolio credit loss model evaluated by Huang et al. (2007). In the next two chapters those models are compared and their efficiency and characteristics are discussed.

1. Normal Approximation (NA) Method

This method is derived for the first time by Martin in 2004. Most simply put, it is just an application of the Central Limit Theorem (CLT).

***CLT**: Under some conditions the average (or sum) of a large number of independent random variables is normally distributed.*

One of the conditions for the CLT to hold is that the random variables should be identically distributed. In our case that condition is not satisfied, but actually it is required only to make the proof feasible. In general, the CLT applies to the case of non-identical distributions as long as the set of distributions is bounded in terms of mean and variance (for details - WolframMathworld). Simply put, the CLT applies for any distribution of finite variance and in our case the conditional variance used below fulfils the criterion. We now would like to use the CLT to approximate the portfolio loss L conditional on the common factor Y. Let us assume that it is normally distributed with mean

$$\mu(Y) = \sum_{i=1}^{n} w_i p_i(Y)$$

and variance

$$\sigma^2(Y) = \sum w_i^2 p_i(Y)(1 - p_i(Y)).$$

The variables $w_i = EAD_i \times LGD_i$ and $p(Y) = P[D_i = 1 | Y]$ are given as in the Vasicek model in chapter I, part 1. We can now easily express the conditional tail probability by:

$$P[L > x | Y] = \Phi\left(\frac{\mu(Y) - x}{\sigma(Y)}\right)$$

and thus by integrating over Y we can derive the unconditional tail probability:

$$P[L > x] = E_Y\left[\Phi\left(\frac{\mu(Y) - x}{\sigma(Y)}\right)\right] = \int_x^{\infty} \Phi\left(\frac{\mu(y) - x}{\sigma(y)}\right)\phi(y)dy.$$

In the above equations Φ denotes the cumulative distribution function and ϕ denotes the probability density function of the standard normal distribution. In order to numerically approximate the integral and calculate the required VaR we consider the so called Discrete Fourier Transformation (DFT), which in general turns the integral into a finite sum that is calculated instead. For more details on the discretisation procedure refer to Martin (2004).

In order to calculate the VaRC of a given obligor i we need to compute the derivatives of the tail probability $P[L > x]$, of the conditional mean $\mu(Y)$ and of the conditional variance $\sigma^2(Y)$ of the portfolio loss L with respect to the asset allocations w_i. Those derivatives are given by the expressions:

$$\frac{\partial}{\partial w_i} P[L > x] = E_Y\left[\left(\frac{1}{\sigma(Y)}\frac{\partial \mu(Y)}{\partial w_i} - \frac{1}{\sigma}\frac{\partial x}{\partial w_i} - \frac{\mu(Y) - x}{\sigma^2(Y)}\frac{\partial \sigma(Y)}{\partial w_i}\right)\phi\left(\frac{\mu(Y) - x}{\sigma(Y)}\right)\right],$$

$$\text{with } \frac{\partial \mu(Y)}{\partial w_i} = p_i(Y) \text{ and } \frac{\partial \sigma(Y)}{\partial w_i} = \frac{w_i p_i(Y)(1 - p_i(Y))}{\sigma(Y)}.$$

We want the tail probability $P[L > VaR_\alpha]$ to be fixed at $1-\alpha$ and $VaR_\alpha = x$ to vary. This means that the left hand-side of the above equation vanishes for $VaR_\alpha = x$:

$$0 = \frac{\partial}{\partial w_i}(1-\alpha) = E_Y\left[\left(\frac{1}{\sigma(Y)}\frac{\partial \mu(Y)}{\partial w_i} - \frac{\mu(Y) - VaR_\alpha}{\sigma^2(Y)}\frac{\partial \sigma(Y)}{\partial w_i}\right)\phi\left(\frac{\mu(Y) - VaR_\alpha}{\sigma(Y)}\right)\right] - E_Y\left[\frac{1}{\sigma}\frac{\partial VaR_\alpha}{\partial w_i}\phi\left(\frac{\mu(Y) - VaR_\alpha}{\sigma(Y)}\right)\right] \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{\partial VaR_\alpha}{\partial w_i} = \frac{E_Y\left[\left(\frac{1}{\sigma(Y)}\frac{\partial \mu(Y)}{\partial w_i} - \frac{\mu(Y) - x}{\sigma^2(Y)}\frac{\partial \sigma(Y)}{\partial w_i}\right)\phi\left(\frac{\mu(Y) - x}{\sigma(Y)}\right)\right]}{E_Y\left[\frac{1}{\sigma}\phi\left(\frac{\mu(Y) - x}{\sigma(Y)}\right)\right]} =$$

$$= \frac{E_Y\left[\left(\frac{1}{\sigma(Y)}p_i(Y) - \frac{\mu(Y)-x}{\sigma^2(Y)}\frac{w_i p_i(Y)(1-p_i(Y))}{\sigma(Y)}\right)\phi\left(\frac{\mu(Y)-x}{\sigma(Y)}\right)\right]}{E_Y\left[\frac{1}{\sigma}\phi\left(\frac{\mu(Y)-x}{\sigma(Y)}\right)\right]}.$$

We finally derive the VaRC for obligor i to be: $VaRC_{i,\alpha} = w_i \dfrac{\partial VaR_\alpha}{\partial w_i}$.

2. Saddlepoint Approximation (SA) Method

The saddlepoint approximation (SA) method is based on the existence of the so called analytic Moment Generating Function (MGF), which we use to generate the loss distribution of the credit portfolio and is known to approximate small tail probabilities accurately.

Moment Generating Function: For a given random variable L_i (which again represents the loss of asset i) we define the MGF as the unknown analytic function $M_{L_i}(t) = E[e^{tL_i}]$. If $i = 1,\ldots,n$ and $L = \sum_{i=1}^{n} L_i$ is the total portfolio loss, then the MGF of L is given by:

$$M_L(t) = \prod_{i=1}^{n} M_{L_i}(t) = E[\exp(t\sum_{i=1}^{n} L_i)] = \int e^{tx} f_L(x)dx,$$

with f_L being the density function of L.

Cumulant Generating Function: We introduce also the Cumulant Generating Function (CGF) of L by simply taking the logarithm of the MGF: $K_L(t) = \log(M_L(t))$. We use it to express the inverse MGF of L and thus its distribution, also known as the Bromwich integral (or the Fourier-Mellin integral), which we derive using Fourier inversion (for details refer to Taras (2005)):

$$f_L(x) = \frac{1}{2j\pi}\int_{-j\infty}^{+j\infty} e^{-tx} M_L(t)dt = \frac{1}{2j\pi}\int_{-j\infty}^{+j\infty} e^{-tx}\exp(K_L(t))dt = \frac{1}{2j\pi}\int_{-j\infty}^{+j\infty}\exp(K_L(t) - tx)dt,$$

where $j = \sqrt{-1}$.

The Direct Saddlepoint Approximation: The use of saddlepoint approximation for credit portfolios appeared in the literature for the first time around 2001. In the beginning it was applied by Martin (2001a, b) to the unconditional MGF of the portfolio loss L. The MGF is unique, thus the Bromwich integral exists, but in general it is not analytically tractable. Traditional approaches for solving it lead to ill-conditioning and truncation errors. Thus saddlepoint approximation (SA) arises in this setting to give an accurate analytic alternative for f_L which avoids calculating the integral. For detailed description of saddlepoint approximations refer to Jensen (1995). Let $\tilde{t}$ be the point at which $K_L(t) - tx$ is stationary. It is then called the saddlepoint and is given by the solution of $K'_L(\tilde{t}) = x$ (the derivative is with respect to t).

The main idea of the method is that the density function f_L and the unconditional tail probability $P[L > x]$ can be approximated by the CGF and its derivatives at $\tilde{t}$ up to some order. For that purpose we use Taylor expansion of the function $K_L(t) - xt$ as a function of t around the saddlepoint $\tilde{t}$:

$$K_L(t) - tx = K_L(\tilde{t}) - \tilde{t}x + 1/2(t - \tilde{t})^2 K''_L(\tilde{t}) + 1/3(t - \tilde{t})^3 K'''_L(\tilde{t}) + \ldots.$$

The n-th derivative of the MFG is given by $M_L^{(n)}(t) = E[L^n e^{tL}]$ and thus:

$$K'_L(t) = \frac{M'_L(t)}{M_L(t)}; \quad K''_L(t) = \frac{M_L(t)M''_L(t) + M'_L(t)^2}{M_L(t)^2},$$

and so on. There are a few different approximations that have been studied already, depending on the order of the Taylor expansion. The formula for the density we consider is derived by Daniels (1987) and the one for the tail probability - by Lugannani & Rice (1980). Both of them use fourth-order Taylor expansion.

These formulae are given by:

$$f_L(x) \approx \frac{\phi(z_l)}{\sqrt{K''_L(\tilde{t})}} \left[1 + \left(-\frac{5K'''_L(\tilde{t})^2}{24K''_L(\tilde{t})^3} + \frac{K_L^{(4)}(\tilde{t})}{8K''_L(\tilde{t})^2} \right) + O(n^{-2}) \right]$$

$$P[L > x] \approx 1 - \Phi(z_l) + \phi(z_l)\left(\frac{1}{z_w} - \frac{1}{z_l} + O(n^{-3/2})\right)$$

with $z_l = \operatorname{sgn}(\tilde{t})\sqrt{2[x\tilde{t} - K_L(\tilde{t})]}$ and $z_w = \tilde{t}\sqrt{K''_L(\tilde{t})}$.

The Indirect Saddlepoint Approximation: In the previous part we applied the SA to the unconditional MGF of the portfolio loss L, but the assumption of mutual independence of all L_i is then violated. Annaert (2006) shows that in this case the method is inefficient and the results are wrong for portfolios with exposure size of high skewness and kurtosis. Despite those cons the model is simple and can be easily extended to the one we mention in the next part - the Simplified Saddlepoint Approximation (SSA). To avoid the shortfalls of the Martin's method, Huang (2006) applies the SA method to the conditional MGF of the portfolio loss L under Y. Then all the separate losses L_i are indeed independent. The latter approach is called Indirect SA and is mostly used in practice at the cost of some brute computations – for every realization of the common factor Y we need to estimate the saddlepoint $\tilde{t}$ from the equation $K'_L(\tilde{t}) = x$.

In line with the previous paragraph Huang (2006) defines the conditional MGF of L to be:

$$M_L(t,Y) = \prod_{i=1}^{n}(1 - p_i(Y) + p_i(Y)e^{w_i t}).$$

Now the conditional CGF and its first four derivatives can be defined as:

$$K_L(t,Y) = \sum_{i=1}^{n}\log(1 - p_i(Y) + p_i(Y)e^{w_i t}),$$

$$K'_L(t,Y) = \sum_{i=1}^{n}\frac{w_i p_i(Y)e^{w_i t}}{1 - p_i(Y) + p_i(Y)e^{w_i t}},$$

$$K''_L(t,Y) = \sum_{i=1}^{n}\frac{(1 - p_i(Y))w_i^2 p_i(Y)e^{w_i t}}{[1 - p_i(Y) + p_i(Y)e^{w_i t}]^2},$$

$$K_L'''(t,Y)=\sum_{i=1}^{n}\left[\frac{(1-p_i(Y))w_i^3 p_i(Y)e^{w_i t}}{[1-p_i(Y)+p_i(Y)e^{w_i t}]^2}-\frac{2(1-p_i(Y))w_i^3 p_i^2(Y)e^{2w_i t}}{[1-p_i(Y)+p_i(Y)e^{w_i t}]^3}\right],$$

$$K_L^{(4)}(t,Y)=\sum_{i=1}^{n}\left[\frac{(1-p_i(Y))w_i^4 p_i(Y)e^{w_i t}}{[1-p_i(Y)+p_i(Y)e^{w_i t}]^2}-\frac{6(1-p_i(Y))w_i^4 p_i^2(Y)e^{2w_i t}}{[1-p_i(Y)+p_i(Y)e^{w_i t}]^3}+\frac{6(1-p_i(Y))w_i^4 p_i^3(Y)e^{3w_i t}}{[1-p_i(Y)+p_i(Y)e^{w_i t}]^4}\right].$$

Once we have calculated the conditional CGF $K_L(t,Y)$ and its derivatives we may use saddlepoint approximation to estimate the conditional density and tail probability of the portfolio loss as explained in the direct SA. For $x\in[0,\sum w_i]$ we have a unique saddlepoint $\tilde{t}$. We get the unconditional loss density and tail probability by integrating over Y (for example $P[L>x]=\int P[L>x\,|\,Y]dP[Y]$). We can calculate the VaR_α by inverting the loss distribution.

To calculate $VaRC_{i,\alpha}$ Huang (2006) differentiates the tail probability with respect to the effective exposures w_i:

$$\frac{\partial}{\partial w_i}P[L>x]=E_Y\left[\frac{1}{2j\pi}\int_{-j\infty}^{+j\infty}\left[\frac{1}{t}\frac{\partial K_L(t,Y)}{\partial w_i}-\frac{\partial x}{\partial w_i}\right]\exp(K_L(t,Y)-tx)dt\right].$$

Just like for the normal approximation in the previous part, if we replace x with VaR_α in the above formula, the left hand-side becomes zero, because the tail probability $P[L>VaR_\alpha]=1-\alpha$. That is how we finally derive the $VaRC_{i,\alpha}$:

$$0=E_Y\left[\frac{1}{2j\pi}\int_{-j\infty}^{+j\infty}\left[\frac{1}{t}\frac{\partial K_L(t,Y)}{\partial w_i}-\frac{\partial VaR_\alpha}{\partial w_i}\right]\exp(K_L(t,Y)-tVaR_\alpha)dt\right]\Leftrightarrow$$

$$\Leftrightarrow VaRC_{i,\alpha}=w_i\frac{\partial VaR_\alpha}{\partial w_i}=w_i\frac{E_Y\left[\int_{-j\infty}^{+j\infty}\left[\frac{\partial K_L(t,Y)}{\partial w_i}\frac{1}{t}\exp(K_L(t,Y)-tVaR_\alpha)dt\right]\right]}{E_Y\left[\int_{-j\infty}^{+j\infty}\exp(K_L(t,Y)-tVaR_\alpha)dt\right]}=$$

$$= w_i \frac{E_Y\left[\int_{-j\infty}^{+j\infty} \frac{p_i(Y)\exp(w_i t)}{1-p_i(Y)+p_i(Y)\exp(w_i t)} \exp(K_L(t,Y)-tVaR_\alpha)dt\right]}{E_Y\left[f_L[VaR_\alpha \mid Y]\right]}.$$

If we define the CGF of L conditional on Y and $D_i = 1$ as:

$$\hat{K}_L^i(t,Y) = \log(p_i(Y)e^{w_i t}) + \sum_{j \neq i} \log(1 - p_j(Y) + p_j(Y)e^{w_j t}),$$

then we can simplify the formula for the $VaRC_{i,\alpha}$ to:

$$VaRC_{i,\alpha} = w_i \frac{\partial VaR_\alpha}{\partial w_i} = w_i \frac{E_Y\left[\int_{-j\infty}^{+j\infty} \exp(\hat{K}_L^i(t,Y) - tVaR_\alpha)dt\right]}{E_Y\left[f_L(VaR_\alpha \mid Y)\right]}.$$

Another way to derive the $VaRC_{i,\alpha}$ explicitly is to set $\hat{L}^i = \sum_{j \neq i} L_j = \sum_{j \neq i} w_j D_j$ to be the loss of the portfolio excluding obligor i. Then we get a different formula:

$$VaRC_{i,\alpha} = w_i E[D_i \mid L = VaR_\alpha] = w_i \frac{f[L = VaR_\alpha; D_i = 1]}{f_L[VaR_\alpha]} = w_i \frac{E_Y[f[\hat{L}^i = VaR_\alpha - w_i \mid Y] p_i(Y)]}{E_Y[f_L[VaR_\alpha \mid Y]]}$$

with f being the conditional loss density of the portfolio excluding obligor i, which can also be approximated by direct SA.

The two formulae for the $VaRC_{i,\alpha}$ are the same in the sense that they both are approximated by saddlepoint approximations with the same saddlepoints. The denominators are approximated by the saddlepoint derived from $K'_X(\tilde{t}) = x$. The nominators are approximated using the saddlepoint $\tilde{t}_i$ equal to the solution of the following equation:

$$\sum_{\forall k \neq i} \frac{w_k p_k(Y)\exp(w_k t)}{1 - p_k(Y) + p_k(Y)\exp(w_k t)} = VaR_\alpha - w_i .$$

Although we assume heterogeneity of the obligors, it is more efficient to group the obligors into buckets of similar characteristics as much as possible, especially for larger portfolios.

Then the computation of the CGF and its derivatives is more simplified and the number of separate VaRC's is lower.

3. Simplified Saddlepoint Approximation (SSA) Method

The simplification comes from omitting the n calculations of $\tilde{t}_i$ for every realization of Y. Instead we calculate $\tilde{t}$ only. In general this is possible if we assume that both values are close to each other, thus saving significant amount of calculation time. Of course in some cases the model is less accurate (if the assumption is strongly violated). Basically this method was first presented by Martin (2001b) and then further developed during the next years. Martin assumed that all obligors are independent and derived an explicit formula for the value at risk contribution of obligor i with $K_L(t)$ being the CGF of the portfolio loss L:

$$VaRC_{i,\alpha} = \frac{w_i}{\tilde{t}} \frac{\partial K_L(t)}{\partial w_i}\Big|_{t=\tilde{t}} = \frac{w_i p_i \exp(w_i \tilde{t})}{1 - p_i + p_i \exp(w_i \tilde{t})}.$$

If the obligors are conditionally independent instead of independent we can extend the model and derive the SSA solution, in which $f_L(VaR_\alpha | Y)$ should be computed by direct SA:

$$VaRC_{i,\alpha} \approx \frac{E_Y\left[f_L(VaR_\alpha | Y) \frac{w_i p_i(Y) \exp(w_i \tilde{t})}{1 - p_i(Y) + p_i(Y) \exp(w_i \tilde{t})} \right]}{E_Y[f_L(VaR_\alpha | Y)]}.$$

For details about the derivation of the given formula refer to Antonov (2005).

4. Importance Sampling (IS) Method

The method is initially suggested by Glasserman & Li (2005) and Glasserman (2006) as a method for VaR and VaRC calculation, and is later adopted by Huang (2007). This particular IS is a procedure of rare event simulation for credit risk management with main idea to increase the frequency of rare event occurrence by changing the distribution from which we sample. In this framework capturing the dependence between obligors in the portfolio is of great importance. In the later described approach this is accomplished by using the normal (Gauss) copula model, which is widely used by practitioners nowadays.

We assume that the unconditional PD's are given, as well as the effective exposures w_i. In the Gauss copula model the dependence between the obligors is modeled by introducing dependence amongst the default indicators D_i (for details - Glasserman & Li (2005)). Thus the used formula for the conditional default probabilities is the already familiar one from the previous parts (in the case of one common factor):

$$p_i(Y) = p_i[D_i = 1 | Y] = \Phi\left(\frac{\sqrt{\rho_i} Y + \Phi^{-1}(p_i)}{\sqrt{1-\rho_i}}\right).$$

In general the IS method consists of two parts (and is called two-step IS):

- Exponential Twisting – change of the distribution of the conditional PD's in order to increase them and make losses less rare events.
- Mean Shifting – shifting the mean of the common factor Y for the same reason as exponential twisting.

Exponential Twisting: The purpose of exponential twisting is to increase (twist) the conditional default probabilities $p_i(Y)$ to the new probabilities given by the formula below in order to increase the estimates of our tail probabilities and make the defaults less rare:

$$q_{i,\theta(Y)}(Y) = \frac{p_i(Y)\exp(\theta(Y)w_i)}{1 + p_i(Y)[\exp(\theta(Y)w_i) - 1]} \text{ with } \theta > 0.$$

Thus we obtain a new probability measure Q (notice that the higher the exposure w_i the higher the increase in the default probability). We see that for $\theta = 0$ the new probabilities are the same as the old ones and for $\theta > 0$ there is indeed an increase. There are many possibilities to increase the PD's, but this one has specific features that make it more effective. One of them is the resulting form of the likelihood ratio that corrects for changing those probabilities. It is given by:

$$\left(\frac{p_i(Y)}{q_{i,\theta(Y)}(Y)}\right)^{D_i}\left(\frac{1-p_i(Y)}{1-q_{i,\theta(Y)}(Y)}\right)^{1-D_i} = e^{-\theta(Y)D_i w_i}\left(1 + p_i(Y)\left(e^{\theta(Y)w_i} - 1\right)\right).$$

Given that the default indicators D_i are independent under Y, the likelihood ratio for changing all PD's is simply the product of the individual likelihood ratios:

$$\prod_{i=1}^{n} e^{-\theta D_i w_i}\left(1+p_i(Y)\left(e^{\theta w_i}-1\right)\right)=\exp\left(-\theta(Y)L+K(\theta(Y),Y)\right),$$

with $K(\theta(Y),Y)=\log\left(E\left[\exp\left(\theta(Y)L\right)\mid Y=y\right]\right)=\sum_{i=1}^{n}\log\left(1+p_i(Y)\left(e^{\theta w_i}-1\right)\right)$ being the conditional CFG of L. Glasserman & Li (2005) suggest that $\tilde{\theta}(y)$ be the unique solution of the equation (unique solution indeed exists due to the fact that the derivative is monotonically increasing in θ):

$$K'(\theta(y),y)=\sum_{i=1}^{n} q_{i,\theta(Y)}(Y)w_i=x.$$

If $x>E[L\mid Y=y]$ then $\tilde{\theta}(y)>0$ and the new twisted PD's are higher than the original ones. In the case $x\leq E[L\mid Y=y]$ no exponential twisting is needed.

Mean Shifting: In general we have assumed that the common factor Y is standard normally distributed. Let us define a probability measure R under which the common factor Y is normally distributed with mean $\mu\neq 0$ and variance still equal to one (so we have the transition $N(0,1)\rightarrow N(\mu,1)$). The idea is to artificially increase the mean of the distribution and thus also the probability $P[L>x]$, in order to make the losses less rare. After that mean shift, we need to correct for this change of the distribution by multiplying each observation by a given likelihood ratio. Glasserman (2006) states that the ratio is equal to $\exp\left(-\mu Y+1/2\mu^2\right)$ and thus we get the expression for the corrected tail probability under R (given the original PD's):

$$P[L>x]=E^R[1_{\{L>x\}}\exp(-\mu Y+\mu^2/2)].$$

The effectiveness of the IS depends to a great degree on the choice of μ. The value of μ proposed by Glasserman (2006) is the solution $\tilde{\mu}$ (which is unique under the one-factor model) of the equation: $\max_y \{K(\tilde{\theta}(y), y) - \tilde{\theta}(y)x - 1/2y^2\}$ for $\tilde{\theta} \geq 0$.

Calculating the VaRC's: Using the two parts explained above we can derive the tail probability under the new probability measures Q and R (using the new probabilities of default and the new distribution of the common factor Y):

$$P[L > x] = E\left\{E^Q\left[1_{\{L>x\}} \prod_i \left(\frac{p_i(Y)}{q_{i,\theta(Y)}(Y)}\right)^{D_i} \left(\frac{1-p_i(Y)}{1-q_{i,\theta(Y)}(Y)}\right)^{1-D_i} \middle| Y\right]\right\} =$$

$$= E\left\{E^Q\left[1_{\{L>x\}} \exp(-\theta(Y)L + K(\theta(Y), Y) \middle| Y\right]\right\} =$$

$$= E^R\left\{\exp(-\mu Y + \mu^2/2) E^Q\left[1_{\{L>x\}} \exp(-\theta(Y)L + K(\theta(Y), Y) \middle| Y\right]\right\}.$$

In that case the $VaRC_{i,\alpha}$ of obligor i can be directly expressed by the formula:

$$VaRC_{i,\alpha} = w_i \frac{\sum_k D_i l^k 1_{\{L^k = VaR_\alpha\}}}{\sum_k l^k 1_{\{L^k = VaR_\alpha\}}}.$$

In the given formula the subscript k denotes the k-th simulation and l stands for the combined likelihood ratio (a multiplication of the ratios of the shifting and of the twisting):

$$l = \exp[-\mu Y + \mu^2/2 - \theta(Y)L + K(\theta(Y), Y)].$$

V. Numerical Evidence and Comparison of Explained Models

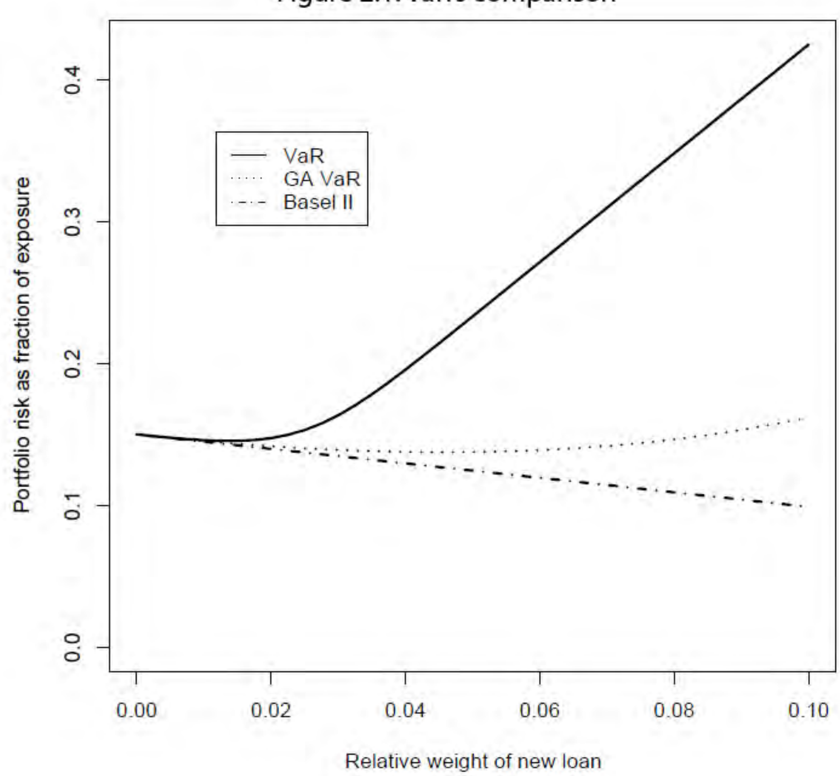
In this part of the paper are presented the numerical results derived by the authors of the described granularity adjustments that will help us to compare those techniques and will prepare us for the next part, where we are going to finally investigate which of those techniques are preferable and in what situations.

1. The Semi-Asymptotic Approach

To demonstrate the model described in part III.4, Emmer & Tasche (referred to as 'the authors' till the end of the chapter) consider a portfolio driven by systematic risk Y with an average credit standing ($E(X) = 0,025 = \Phi(r)$) and asset correlation ρ. The main idea is to enlarge the portfolio by an additional loan, expressed by the indicator D in the equation given before - $PL(c) = cD + (1-c)Y$. This loan has a high credit worthiness - $p = P(ED) = 0,002 = \Phi(a)$ and asset correlation τ.

In line with the Basel II proposal for corporate loan portfolios (BCBS (2005)) the authors set $\rho = 0,154$ and $\tau = 0,229$. They would like to plot the risk (calculated by means of VaR at confidence 99,9%) of the portfolio loss variable $PL(c)$ as defined in the above equation, as a function of the relative weight c of the additional loan. They plot three VaR's for comparison: the true VaR, the granularity adjustment approximation of the VaR and the VaR according to the Basel II Accord (for details on the exact computation steps refer to Emmer (2005)). On Figure 2.1 we see that for w up to 2% both approximations are quite accurate, compared to the true VaR. For higher relative weight of the new loan they start losing quality very fast. The Basel II VaR becomes more and more inaccurate also compared to the granularity adjusted VaR. For example for c of about 4% it is about 7 times more inaccurate. The reason for that sudden drop of quality is of course the assumption of both

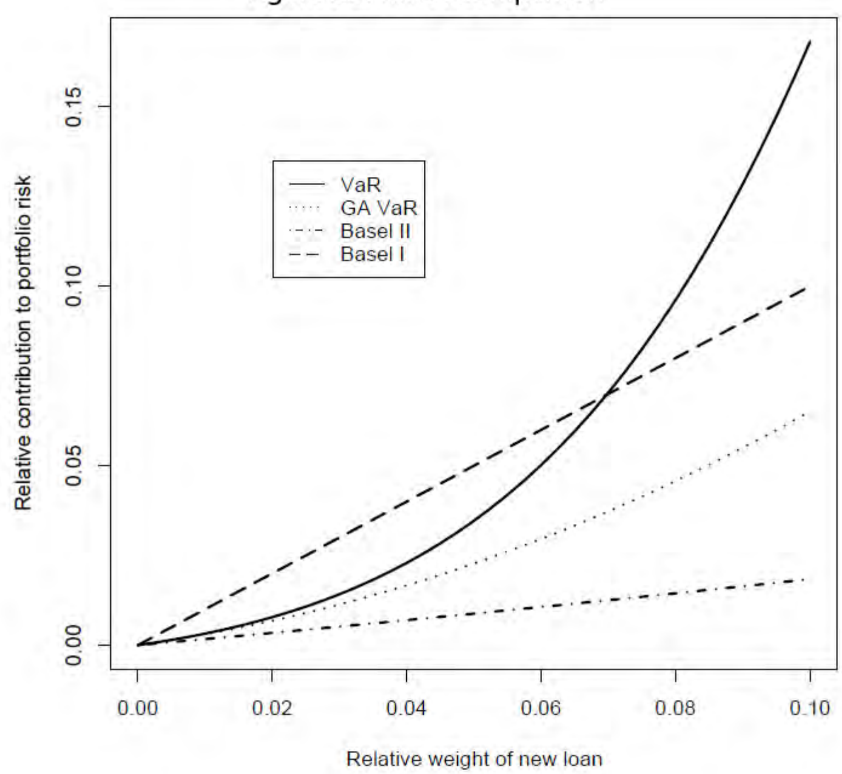
Figure 2.1: VaR's comparison



methods that the portfolio has no name concentration. Obviously, for c higher than 2% this is not the case.

Another aspect that the authors consider is a comparison of relative VaRC's of the new loan to the portfolio risk, considered also as functions of c. They plot on Figure 2.2 those contributions to the true VaR, the GA VaR, as well as to the Basel I and II VaR's. The Basel I curve crosses the real contribution curve at the minimal risk portfolio, for relative weight c^* of about 7%. Again for c up to 2% the GA contribution is quite accurate compared to the real contribution, but for higher weights it starts losing quality fast. The Basel II curve is not a bad approximation of the true VaRC as well for relative weight of the new loan up to 1,5% – 2% (although not as good as the GA VaRC), but after that it becomes way too inaccurate (also compared to the GA VaRC). We also notice that both the GA VaRC and the Basel II VaRC curves lie below the Basel I curve. That may give us the wrong feeling that higher relative weight of the additional loan may still improve the overall risk of the portfolio. To provide a reasonable argument against that claim we take a look again at Figure 2.1 and we notice that for a higher than 7% w the VaR of the portfolio is unacceptably high.

Figure 2.2: VaRC's comparison



2. The Gordy and Lütkebohmert Approach

To demonstrate the granularity adjustment approach described in section III.5 Gordy and Lütkebohmert (referred to as 'the authors until the end of the chapter) apply it to a number of realistic bank portfolios. The data used is provided by the German credit register and consists of loans greater than or equal to 1,5 Million Euros. The portfolios they use contain more than 250 exposures and they believe that is an appropriate size for calculating the GA. Those portfolios are divided into 4 groups, according to the number of exposures and degree of heterogeneity: large (more than 4000 exposures), medium (1000 - 4000 exposures), small (600 - 1000 exposures) and very small (250 - 600). The PD of each loan is predetermined by the S&P's credit ratings.

The authors calculate the GA as percentage of the total portfolio exposure for the given portfolio groups (the results are given on Figure 3.1) and compare it to the HHI. We notice that for the large portfolios the GA is 12 to 14 basis points, but it grows substantially for the very small portfolios up to 161 basis points. Also a high correlation with the HHI is observable. The correlation is not perfect though, due to the fact that the HHI is not sensitive to the PD of the exposures (and thus to the credit ratings). In their results the authors also include a benchmark (reference) portfolio of 6000 loans each with $PD = 0{,}01$ and $ELGD = 0{,}45$, and of homogeneous EAD. We notice the importance of heterogeneity for calculating the GA by comparing the GA for the benchmark and for the largest portfolio. Although the number of exposures of both portfolios is big enough, the GA of the benchmark portfolio is about six times smaller due to its homogeneity in name concentrations.

Figure 3.1: Granularity Adjustment for Real Bank Portfolios

Portfolio	Number of Exposures	HHI	GA (in %)
Reference	6000	0.00017	0.018
Large	> 4000	< 0.001	$0.12 - 0.14$
Medium	$1000 - 4000$	$0.001 - 0.004$	$0.14 - 0.36$
Small	$600 - 1000$	$0.004 - 0.011$	$0.37 - 1.17$
Very Small	$250 - 600$	$0.005 - 0.015$	$0.49 - 1.61$

Another thing Gordy and Lütkebohmert calculate is the VaR of the large and medium portfolios (by means of the CreditRisk^{+} model) and the GA as percentage of this VaR. For the large portfolios the GA add-on is between 3% and 4% and for the medium ones it is between 5% and 8%. That is evidence that GA is higher for smaller portfolios as percentage of the VaR.

Next the authors compute the GA for the large, medium and small portfolio groups as percentage of the Basel II Risk Weighted Assets (RWA), used for estimation of IRB capital charges. The results are given in Figure 3.2. As shown in Figure 3.1 the GA for the benchmark portfolio is about 0,018%. The IRB capital charge for this same portfolio (the capital requirements that account for the systematic risk of the portfolio) is 5,86% of the total portfolio exposure. Thus the authors derive that the relative add-on due to granularity is about 0,3% of the RWA. Same methodology is used to calculate the add-on for the three real portfolio groups. Again the heterogeneity of the large bank portfolios plays important role in calculating the GA – the GA for the large portfolios is more than 13 times higher than that of the reference portfolio. Nevertheless, the GA is quite small for the large portfolios and also for some of the medium ones, but highly significant for the small portfolios (up to 32% of the RWA).

Figure 3.2: Granularity Adjustment as percentage of RWA

Portfolio	Number of Exposures	Relative Add-On for RWA
Reference	6000	0.003
Large	> 4000	0.04
Medium	$1000 - 4000$	$0.04 - 0.10$
Small	$300 - 1000$	$0.17 - 0.32$

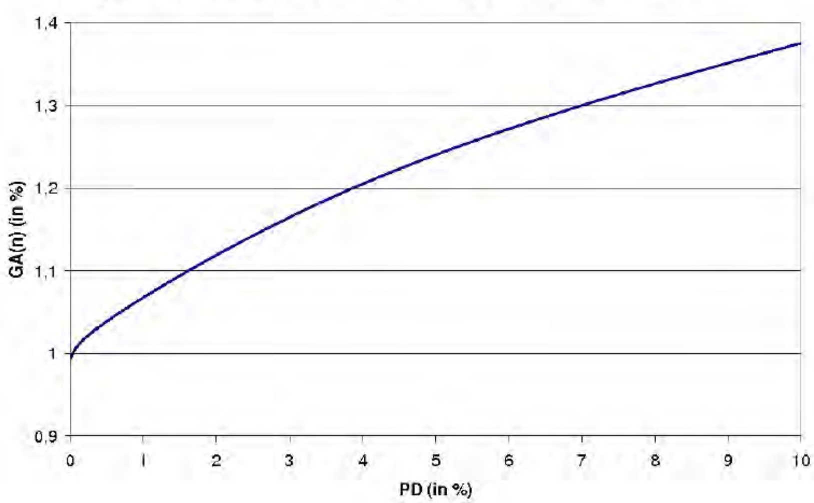
Another important issue is to investigate the approximation quality of the simplified GA ($\tilde{GA}$) compared to the 'full' GA. For that reason Gordy and Lütkebohmert consider six portfolios of 1000 exposures each, constant PD (either 1% or 4%) and $ELGD = 0{,}45$. The portfolios differ by degree of heterogeneity – P0 is completely homogeneous and P50 is with high name concentration (the largest exposure $A_{1000} = 1000^{50}$ accounts for 5% of the total exposure). The results are given on Figure 3.3 below. In general we notice that the error increases with concentration and PD, but it is really trivial for realistic portfolios. Even for P10 and PD of 4% the error is 3,2 basis points and for the extreme case P50 with PD=4% the error is 14,1 basis points, but both are small compared to the total GA. That proves the claim that the simplified GA is a good approximation that may be used in practice.

Figure 3.3: Comparison between GA and Simplified GA

Portfolio	P0	P1	P2	P10	P50
PD = 1%					
Exposure A_i	1	i	i^2	i^{10}	i^{50}
$\widetilde{\mathrm{GA}}$ in %	0.107	0.142	0.192	0.615	2.749
GA in %	0.109	0.146	0.197	0.630	2.814
PD = 4%					
Exposure A_i	1	i	i^2	i^{10}	i^{50}
$\widetilde{\mathrm{GA}}$ in %	0.121	0.161	0.217	0.694	3.102
GA in %	0.126	0.168	0.227	0.726	3.243

As stated earlier, one major drawback of ad hoc GA methods is the fact that they do not account for the PD's of the loans, but that is not the case with the model introduced by Gordy and Lütkebohmert. Figure 3.4 shows the dependence of the simplified GA on the default probability. The GA is calculated in percentage of the total portfolio exposure for homogeneous portfolios of 100 borrowers with different PD's. We notice that for increase of the PD from zero to 10% the GA becomes about 0,4% higher.

Figure 3.4: Effect of PD (credit rating) on the Simplified GA



In part III.5 we explained that the variance parameter ξ of the systematic risk factor Y may differ according to the calibration method we consider. For that reason the authors explore the effect of different ξ values on the GA. On one hand the GA is increasing with ξ

(for example for ξ going up from 0,2 to 0,3 the GA increases by 10%). On the other hand the absolute degree of the GA is really sensitive to changes in ξ, but the relative one across bank portfolios is in fact not that sensitive at all. That is why we conclude that in general the precision of the calibration of ξ is not that crucial for the efficient functioning of the GA as a supervisory tool.

3. Numerical Methods

In this part we investigate the quality of the methods described in chapter IV, by means of numerical results estimated by Huang, Oosterlee and Mesters (referred to as 'the authors until the end of the chapter). They use a stylized portfolio of low granularity (meaning that the largest obligor has a lot higher exposure than the smallest one), consisting of 11325 obligors. It is divided in six buckets of different exposure size (see Table 4.1). The total exposure of the portfolio is 54000. Exposure concentrations are not significant due to the fact that the weight of the largest obligor is less than 1,5%. The asset correlation ρ is 20% and the PD is 0,33% for each loan.

Bucket:	1	2	3	4	5	6
Exposure:	1	10	50	100	500	800
# of Obligors	10.000	1000	200	100	20	5

Table 4.1: The stylized portfolio used to evaluate numerical models for computation of VaR.

In Huang (2007) may be observed additional details on the computations compared later on. For the rest of the part by 'Vasicek' we denote asymptotic approximation of the Vasicek model. 'NA' and 'SA' stand for the normal and saddlepoint approximations respectively. By 'IS-10K' we denote importance sampling with ten thousand scenarios, which are divided into 10 samples of equal size in order to estimate VaR and sample standard deviations. For a reference (benchmark) method the authors use a Monte Carlo simulation based on ten subsamples with 16 Million scenarios each. By CI we denote the 95% confidence intervals. In Figure 4.1 we compare both 99,9% and 99,99% VaR's calculated by the above mentioned methods. Computation (CPU) time is reported in seconds. In brackets next to the VaR estimates we show also the sample standard deviations. The main thing we notice by first looking at the results is the fact that even for the stylized portfolio of low exposure concentration the Vasicek results at both confidence levels are far from the benchmark results (relative errors about 5%). The normal approximation outputs are better at the cost of small additional computation time (the relative errors are less than 1%). The saddlepoint approximation results are even better (they are both in the 95% confidence interval of the benchmark and the relative errors are less than 0,2%) and the method remains pretty fast. The results from the importance sampling are quite similar to those of the SA, but the computational costs are significantly higher.

Figure 4.1: VaR Comparison of Numerical Methods

	$VaR_{99.9\%}$	$VaR_{99.99\%}$	time
Benchmark 95% CI	3960.3(7.7) [3945.2, 3975.3]	6851.6(38.4) [6776.3, 6926.9]	
Vasicek	3680.5	6477.0	8E-4
NA	3924	6804	2E-2
SA	3965	6841	6E+0
IS-10K	3975.3(56.4)	6836.8(84.9)	2E+3

For all the numerical methods the authors estimate the VaRC of an obligor in a slightly different manner than explained in chapter IV – scaled by the effective weight of the obligor (thus the VaRC's are expressed as percentage of the obligor's effective exposure and the values are all between 0 and 1), i.e.

$$\frac{\partial VaR_{\alpha}}{\partial w_i}(L) = P[c_i D_i = 1 \mid L = VaR].$$

On Figure 4.2 we give the VaRC's for each bucket at two different loss levels - $L = 4000$ and $L = 6800$. We see that the Vasicek approximation simply does not work for calculating the VaRC's and no results fall into the 95% confidence interval. The normal approximation is also quite inaccurate with 7 estimates outside the confidence interval and 1,27% difference with the benchmark. Importance sampling has 5 outcomes outside the confidence interval and 1,24% difference with the benchmark. A good method here is the simplified saddlepoint approximation with only two results outside the confidence interval and 1,14% difference with the benchmark. An issue with IS is that the calculated contributions do not increase monotonically with the effective weight w and for that reason we conclude that 10000 scenarios are not enough. The most accurate model is the full saddlepoint approximation with all estimates inside the confidence interval and a maximal error of 0,33%. On the other hand, the SSA is about seven times faster than the full SA.

Figure 4.2: VaRC Comparison of Numerical Methods

(a) VaRC at the loss level $L = 4000$

	VaRC1	VaRC2	VaRC3	VaRC4	VaRC5	VaRC6	time
Benchmark 95% CI	6.33%(0.04%) [6.25%, 6.41%]	6.38%(0.05%) [6.28%, 6.48%]	6.54%(0.03%) [6.49%, 6.59%]	6.86%(0.08%) [6.70%, 7.02%]	9.36%(0.17%) [9.02%, 9.70%]	11.32%(0.38%) [10.58%, 12.06%]	
Vasicek	7.41%	7.41%	7.41%	7.41%	7.41%	7.41%	3E-3
NA	6.55%	6.59%	6.78%	7.02%	8.92%	10.35%	1E-2
SA	6.35%	6.39%	6.58%	6.82%	9.21%	11.65%	2E+0
SSA	6.35%	6.37%	6.5%	6.68%	9.12%	12.46%	3E-1
IS-10K	6.54%	6.46%	6.77%	6.7%	9.4%	10.33%	1E+3

(b) VaRC at the loss level $L = 6800$

	VaRC1	VaRC2	VaRC3	VaRC4	VaRC5	VaRC6	time
Benchmark 95% CI	11.23%(0.09%) [11.06%, 11.41%]	11.29%(0.09%) [11.11%, 11.48%]	11.56%(0.11%) [11.35%, 11.77%]	11.87%(0.12%) [11.63%, 12.11%]	14.89%(0.21%) [14.48%, 15.30%]	17.86%(0.59%) [16.70%, 19.03%]	
Vasicek	12.59%	12.59%	12.59%	12.59%	12.59%	12.59%	3E-3
NA	11.42%	11.48%	11.74%	12.06%	14.65%	16.59%	1E-2
SA	11.23%	11.29%	11.55%	11.88%	14.94%	17.78%	2E+0
SSA	11.23%	11.27%	11.48%	11.75%	14.89%	18.44%	3E-1
IS-10K	11.34%	11.52%	11.62%	12.03%	14.83%	16.62%	1E+3

As stated earlier, the portfolio considered until now had no serious name concentration. We now proceed with another testing portfolio used by Huang, Oosterlee and Mesters, for which that is not the case. It consists of one bucket of 1000 obligors with exposure $w_1 = 1$ and a second bucket of only one obligor with a higher exposure $w_2 = S$ (the authors consider two cases - $S = 20$ and extreme concentration $S = 100$). We keep $\rho = 20\%$ and $PD = 0{,}33\%$. As a benchmark is used Binomial Expansion Method introduced in Huang (2006). To investigate the accuracy of the models in case of high name concentration the authors compare the 99,99% VaR in terms of relative error with the benchmark. The results are provided in Figure 4.3. For $S = 20$ we notice that all methods except the Vasicek approximation give relatively small errors below 2%. Things change dramatically in the case $S = 100$: the Vasicek and normal approximations are highly inaccurate, while the SA and IS methods remain within the 2% error bound. The SA is also tested for S up to 1000 and the results

Figure 4.3: VaR Comparison of Numerical Methods for Portfolio with High Concentration S

	$S = 20$				$S = 100$			
	VaR	std	error	time	VaR	std	error	time
Benchmark	125				170			
Vasicek	122.3		-2.13%	6E-4	131.9		-22.39%	1E-3
NA	125		0.00%	1E-2	149		-12.35%	9E-3
SA	126		0.80%	3E+0	168		-1.18%	3E+0
IS-10K	124.1	1.7	-0.72%	2E+2	170.5	3.1	0.29%	2E+2

are still smaller than 2%. That holds also for the SSA, which is not included in the results, but is as accurate as the SA and much faster as well.

For the latter portfolio the authors also investigate VaRC approximations and in Figure 4.4 we present the results both for a small (VaRC1) and for a large (VaRC2) obligors; $S = 20$ and $S = 100$ are considered again. Again the results of the Vasivek method are far from the reality. The NA gives relatively accurate results for $S = 20$, but highly inaccurate ones for $S = 100$. SA is accurate for VaRC1, but not really accurate for VaRC2 and $S = 100$ (the absolute error exceeds 2%). The SSA provides similar results to SA for VaRC1, but for VaRC2 the error is too high (above 5%). The best method here is the IS, which gives absolute errors below 2% in all cases.

Figure 4.4: VaRC Comparison for small (VaRC1) and large (VaRC2) obligors and different exposures S

(a) $S = 20$

$S = 20$	VaRC1	error	VaRC2	error	time
Benchmark	12.06%		21.78%		
Vasicek	12.25%	0.19%	12.25%	-9.53%	3E-3
NA	12.12%	0.06%	18.94%	-2.84%	1E-2
SA	12.05%	-0.01%	21.70%	-0.08%	8E-1
SSA	11.96%	-0.10%	27.06%	5.28%	3E-1
IS-10K	12.04%	-0.02%	22.89%	1.11%	1E+3

(b) $S = 100$

$S = 100$	VaRC1	error	VaRC2	error	time
Benchmark	8.29%		87.07%		
Vasicek	15.45%	7.16%	15.45%	-71.62%	3E-3
NA	12.68%	4.39%	43.18%	-43.89%	2E-1
SA	8.89%	0.60%	90.79%	3.72%	8E-1
SSA	9.15%	0.86%	78.52%	-8.55%	3E-1
IS-10K	8.12%	-0.17%	88.85%	1.78%	1E+3

VI. Efficiency Evaluation of Explained Models

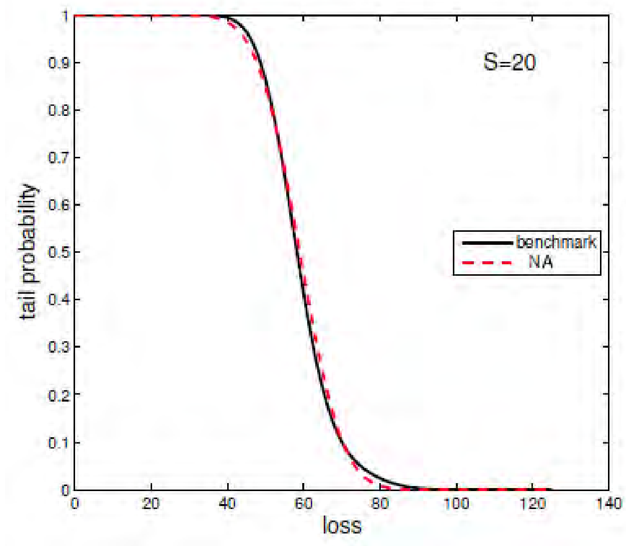
In this final part we are going to talk about the efficiency and robustness of the methods described before and to mention their pros and cons.

The semi - asymptotic approach explained in section III.4 described by Emmer and Tasche in 2005 in general is an extension of the approach introduced by Gordy (2003) that accounts for the exact weights of the portfolio assets. Approximation of the capital charge for every individual asset may be calculated by estimating the partial derivatives of the portfolio loss quantile and multiplying them by the asset weights. The approach also accounts for concentrations, but the results are counter-intuitive in case of very small probability of default of the considered loan. This drawback can be avoided by using coherent risk measures like Expected Shortfall.

The method introduced by Gordy and Lütkebohmert in 2007 is a revision (some technical advances and modifications) and extension of the methodology introduced in the Basel II Second Consultative Paper. After thorough studies of the model on various portfolios, there are two further potential sources of inaccuracy that we need to mention. The first one is the fact that the GA formula is asymptotic and due to this fact it may not work properly for small portfolios. In general it overstates the effect of granularity for that type of portfolios, but is quite accurate for medium-sized portfolios of like 200 low-quality obligors or even 500 investment-grade obligors. The second and more serious issue with this approach is the form of 'basis risk' (or 'model mismatch') – the IRB formulae are based on a different credit risk model. On the other hand the model under consideration has a great advantage – its analytical tractability that permits us to re-parameterize the GA formula in terms of the IRB capital charges, including maturity adjustment. The maturity adjustment of the GA is indirect, in the sense that only the inputs are adjusted, rather than the formula itself. A last limitation of the approach described in section III.5 is the fact that it assumes that each position in the portfolio is an unhedged loan to a single borrower and that makes it difficult to incorporate credit default swaps and loan guarantees in the GA.

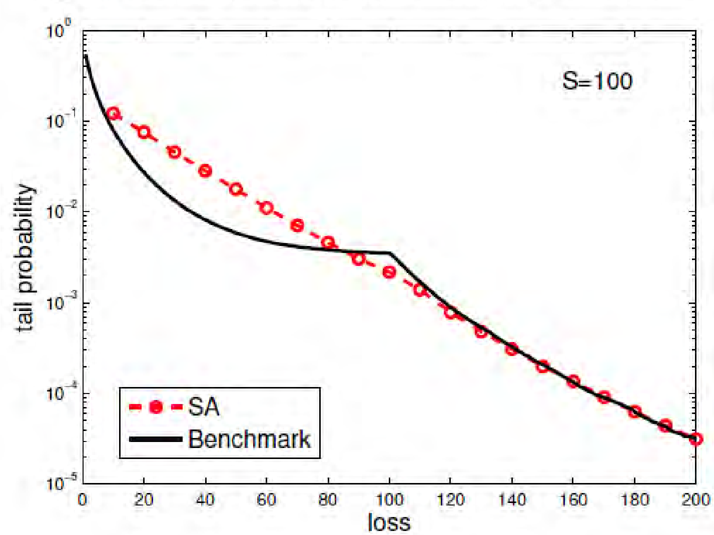
As mentioned earlier the normal approximation described in section IV.1 is based on the central limit theorem. In that case we need to be concerned whether it holds for portfolios with severe exposure concentrations. Apparently that is not the case as we already saw in the previous part where we considered the numerical examples of Huang et al. (2007) (called the authors till the end of the chapter). The reason is simple – the method tries to approximate the loss density by the normal distribution, but it cannot capture the pattern and therefore the CLT does not hold in more extreme cases (concentration of more than 2% of the total exposure). A proof is given by the authors on Figure 5.1 where they compare the tail probability given by the normal approximation conditional on the common factor Y.

Figure 5.1: Comparison of Tail Probabilities Given by NA and Benchmark



Let us now consider the GA from section IV.2. An important issue to handle is to investigate how the saddlepoint approximation can handle name concentrations. To do that, the authors compare the whole loss distribution derived by the SA in case of very high concentration to the benchmark one. We easily notice that the true distribution is not smooth around $S = 100$ (refer to Figure 5.2). Recall that the SA is based on the formulation of the Bromwich integral mentioned before, which represents a probability density function. The portfolio loss L is discrete when the LGD is constant. Then the authors assume that L can be closely approximated by a continuous random variable with absolutely continuous cumulative distribution function. That is how a smoothed version of the loss distribution is produced. On Figure 5.2 we notice that the SA to the tail probabilities is incorrect for almost all quantiles before the non-smoothness one (around 99,6%), but is accurate for higher

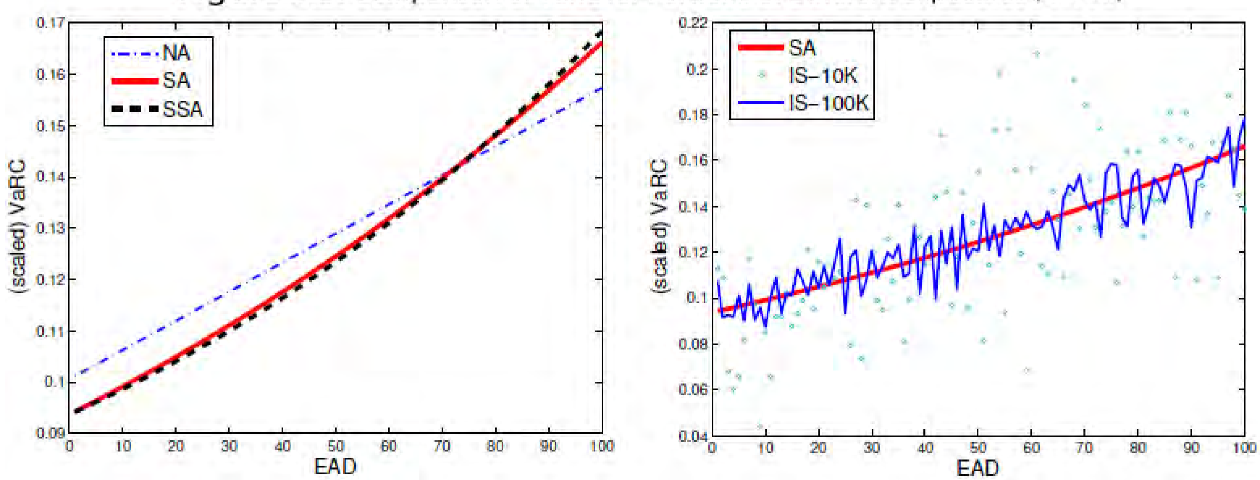
Figure 5.2: Comparison of Loss Distribution by SA and Benchmark



quantiles. In that sense we may conclude that with one or more exceptional exposures in the portfolio the SA may not achieve uniform accuracy of the loss distribution and this may be a problem if the quantile we are interested in precedes the non-smoothness one. As a solution to this problem, Beran (2005) suggests the so called Adaptive Saddlepoint Approximation.

The importance sampling seems to perform well for determining the VaRC's. A reason for that is that in the sample portfolios considered in the previous part, the obligors in each bucket are assumed identical and as a result the calculated VaRC's are less volatile. In general IS can cluster the simulated losses around the required quantile and thus there is a substantial increase in the probability $P[L = VaR]$. On the other hand, quite a large number of replications is needed to obtain such results. Huang et al. consider a portfolio of a hundred obligors, each with different exposure ($w_i = i$, $i = 1,\ldots,100$), assets correlation and PD are again 20% and 0,33%.

Figure 5.3: Comparison of VaRC for Portfolio of Different Exposures (L=700)



On Figure 5.3 the authors show scatterplots of the scaled VaRC at the loss level $L = 700$ (on the y-axis) and the EAD (on the x-axis) for the various numerical methods. All of them provide higher VaRC's for higher EAD, which is highly desirable. Consistent with the results from the last section, the NA overestimates the VaRC of small exposures and underestimates the VaRC of large exposures. The SSA again almost replicates the SA. We notice the difference in the quality of the IS, depending on the number of scenarios. For

10000 scenarios the relation between VaRC and *EAD* is not really clear. For 100000 scenarios we see a huge improvement – the upward trend of VaRC is obvious, but it's highly volatile. For that reason the authors suggest even higher number of simulated scenarios.

As a conclusion about the numerical methods from section IV, we would like to say that for portfolios of lower granularity and medium name concentration all methods provide relatively good solutions. There is no method to be absolutely preferred under all circumstances; the choice will always be a tradeoff between accuracy, robustness and speed. The NA is the fastest method, achieves a fair accuracy and is incapable of handling portfolios with high exposure concentrations. The SSA is the second in speed, performs better than the NA in calculating VaR, but may suffer from the same problem in calculating VaRC's. IS is really robust if a large enough number of scenarios are simulated, because it makes no assumptions on the portfolio structure. The IS VaR results are not always the best amongst the other methods, but are accurate enough. A drawback is the more computational time required for a good estimation of VaRC as stated in the previous paragraph. The SA is more accurate than the NA and the SSA and handles well extreme exposure concentrations. Thus it is a faster substitute of the IS with satisfactory accuracy level, but one should be aware of the non-smoothness of the loss distribution mentioned earlier in this section. Again, the NA and SA are based on asymptotic approximations and thus their accuracy improves significantly by increasing the portfolio size. On the other hand, IS becomes even more time demanding for large portfolios.

VII. Conclusion

In this paper we have dealt with a number of theoretical and practical issues related to the measurement of credit risk and dealing with concentration risk. The numerical results are provided by the authors of the papers that introduced the described Granularity Adjustments and our comparisons and evaluation of the methods are based on those same results. A possible extension of the paper may be the implementation of the models and their comparison on the same portfolios. That will give us more sound and reliable numerical results to compare the techniques and to fully investigate their pluses and drawbacks.

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List of Acronyms:

ASRF –	Asymptotic Single Risk Factor
BCBS –	Basel Committee of Banking Supervision
CGF –	Cumulant Generating Function
CLT –	Central Limit Theorem
DFT -	Discrete Fourier Transformation
EAD –	Exposure at Default
EL –	Expected Loss
GA –	Granularity Adjustment
GBM –	Geometric Brownian Motion
IRB –	Internal Ratings Based
LGD –	Loss Given Default
MGF –	Moment Generating Function
NA –	Normal Approximation
PD –	Probability of Default
RWA –	Risk Weighted Assets
SA –	Saddlepoint Approximation
UL –	Unexpected Loss
VaR –	Value at Risk
VaRC –	Value at Risk Contribution