Valuation of Residential Mortgage-Backed Securities

- A research on valuation approaches -

Master BMI Thesis

Faculty of Applied Sciences

Vrije Universiteit

Mustafa Duru
Supervisor: drs. Dennis Roubos

10-06-2009

Table of Contents

1 Introduction

Financial customers who have bought structured products such as the Mortgage-Backed Securities (hereinafter MBS) and the Collateralized Debt Obligations (hereinafter CDO) had issues with valuation of these products. This was because of the underlying structure which gives rise to complexity. Since those products take a large place in the financial markets, one of the reasons for the emergence of the crisis is that those products have not always been risk free as assumed. The Residential Mortgage-Backed Securities (hereinafter RMBS) in which residential mortgage loans are being passed through to investors in the form of packages, have been assigned the best possible credit ratings by rating agencies. Those ratings have given the impression to the investors that the underlying mortgage loans bring along almost no credit risk. As a result, they have invested without hesitation of credit risk and the accompanying consequences. When early 2007 the first problems began to stand out and each time piled up, the RBMS seemed not to be risk free as assumed. The RMBS with a rating lower than their creditworthiness could not get their part of interest and principal payments. The matter was, in particular, the speed of how the problems piled up and spread over the whole market within a short course of time. Due to the complexity of the products and lack of adequate information, no one could exactly tell where the problems were stated and where they came from. It is expected that the next few years these valuation problems will continue to occur because the context is still complicated from a technical perspective.

As mentioned above, mortgages are used to produce MBS[1]. They are gathered in a pool and this pool of mortgages are sliced in small pieces and finally sold to the investors as packages. We would like to give some brief introduction about mortgages before we go into the MBS. Fabozzi gives the definition of a mortgage in his book *the Handbook of Fixed Income Securities* as follows: “A mortgage loan is a loan secured by the collateral of some specified real estate property, which obliges the borrower to make predetermined series of payments.” A mortgage is a contract between the lender (mortgagee) and the borrower (mortgagor) in which the mortgagee has the right of foreclosure on the loan in case the mortgagor defaults.

Types of real estate properties that can be mortgaged are represented as follows:

[1] Since RMBS is a part of MBS and we restrict us only to the residential variant, for sake of simplicity, we call the residential mortgage-backed securities as MBS through the whole paper.

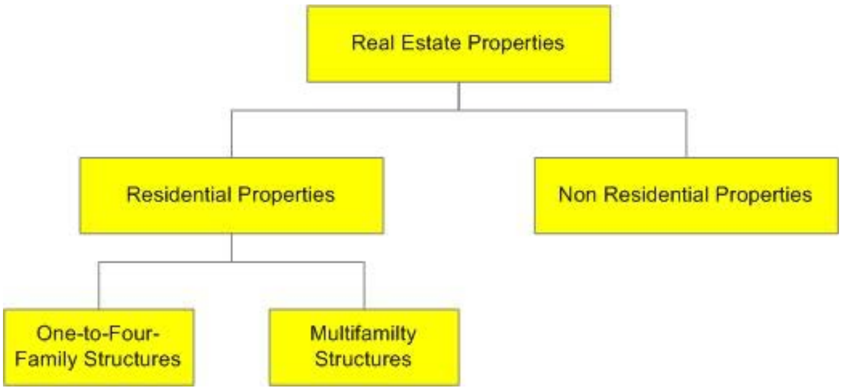

Figure 1: Types of real estate properties

Residential properties include houses, cooperatives and apartments while the non-residential properties include commercial and farm properties etc. We will restrict us to the residential properties since we are interested in valuation of RMBS.

There are several types of mortgage loans that are applicable to residential properties. The most common type is *level payment,* in other words *fixed-rate mortgages.* Other types are *graduated-payment mortgages, growing-equity mortgages, adjustable-rate mortgages, fixed-rate tiered-payment mortgages, balloon mortgages, two-step mortgages* and last but not least *fixed/adjustable-rate mortgage hybrids.* We will restrict us to the most common type of fixed-rate mortgages.

In fixed-rate mortgages the borrower has the obligation to pay a predetermined equal amount on a monthly basis. A monthly amount consists of interest payment and repayment of a portion of the outstanding mortgage balance. Below you find the formula that calculates the monthly mortgage payment [See Fabozzi-1995]:

$$MP = MB_0 \times \frac{[i(1+i)^n]}{[(1+i)^n - 1]}$$ [2]

1-1

where

MP is monthly mortgage payment (€)

MB_0 is original mortgage balance (€)

i is simple monthly interest rate (annual interest rate/12)

n is number of months

[2] See Appendix A for a derivation of this formula

An important issue for the investors of the RMBS is the prepayment risk. Each homeowner has the option to prepay the whole or a part of the outstanding mortgage balance regardless of time to maturity while there is no penalty imposed. This results in uncertainty for the investor because he/she never knows whether the mortgage will be prepaid 1 year or 30 years after the agreement. There are different motives why a mortgagor prepays. Mortgagors prepay the entire mortgage if they sell their home for example for personal reasons. It is also a common practice that the mortgagors refinance a part or the whole if the current mortgage rates fall by a sufficient amount below the contract rate. Refinancing costs should be taken here into consideration. Another possibility is that the mortgager defaults and as a result the collateral is repossessed and sold.

If we assume a pool consisting of thousands of mortgages of which each one bears prepayment risk and as a result each one has different cash flow structures, we can understand how complicated the valuation of such an instrument is. Prepayment models arise here in order to reflect the reality as much as possible. We will discuss this topic entirely later in the upcoming chapters.

The purpose of this thesis is to provide a quantitative approach of how the MBS may be valued. After giving some introduction to the topic, we will go further with the challenging issues such as generic prepayment models, changes in cash flow structures as a result of prepayments and valuation of various models by well-known professors in this area. We will see how they embed (some of) those issues into their valuation models.

2 Prepayment

As mentioned earlier, there are different motives why a mortgagor prepays. In other words, a variety of economic, demographic and geographic factors influence a mortgagor's prepayment decision. The most common factors are interest rates, burnout, seasoning, seasonality, heterogeneity and overall economy[3].

Prepayment is directly related with *interest rates*. As interest rates fall below the mortgage coupon rate, mortgagors will have the incentive to prepay their loan. Historical research shows that not all mortgagors prepay even if it is favourable to do so. Burnout phenomenon arises here.

Heterogeneity of individual mortgagors is closely related to the burnout phenomenon. *Burnout* of a mortgage pool means that the mortgagors will remain in the pool if the interest rates decline below the mortgage coupon rate while others will prepay and leave the pool through refinancing. Those who leave the pool are regarded as being in an interest rate sensitive layer while sitters are regarded as insensitive. In other words, the higher the fraction of the pool that has already prepaid, the less likely are those remaining in the pool to prepay at any interest rate level. This burnout effect causes heterogeneity in the pool.

Seasoning refers to the aging of mortgage loans and can be described as the increasing prepayment incentive of borrowers who are willing to prepay their mortgages if they have not prepaid yet. The Public Securities Association model (also known as "the PSA aging ramp") assumes that prepayment rates increase linearly in the first thirty months of the contract before levelling off.

Seasonality measures the correlation between prepayment rates and the corresponding month of the year. As one might expect, the seasonal increase do occur in the spring and gradually reaches a peak in the late summer and declines in the fall and the winter, with the school year calendar and the weather as the driving forces behind the seasonal cycle.

There has been developed a variety of models to forecast the prepayment behaviour and to represent the prepayment rates and as a result, the timing and the amounts of the cash flows. Prepayment rates tend to fluctuate with interest rates, coupon and age of the underlying mortgage as well as with non-economic factors such as burnout and seasoning. Next section describes different types of prepayment models.

[3] A behaviour of regional or national economy as a whole

2.1 *Twelve-Year Prepaid Life*

This approach has ever been an industry standard. Twelve-year life assumes that there will not be any prepayment during the first twelve year of the mortgage life and then suddenly the mortgage will fully prepay at twelfth year. This approach seems not to give reliable results since prepayment itself tend to fluctuate with interest rates and other factors.

2.2 *Conditional Prepayment Rate (CPR)*

CPR for a given period is the percentage of mortgages outstanding at the beginning of the period that terminates during that period. The CPR is usually expressed on an annualized basis, whereas the term *single monthly mortality* (SMM) refers to monthly prepayment rates [The Handbook of Fixed Income Securities, 1995 - Fabozzi]. More specifically, SMM indicates, for any given month, the fraction of mortgages principal that had not prepaid by the beginning of the month but does prepay during the month. The relationship between the CPR and SMM can be formulated as:

$$CPR = 1 - (1 - SMM)^{12}$$

2-1

An approximation to the above formula is:

$$CPR = 12 \times SMM$$

2-2

In the early years of the mortgage pool, prepayment occurs rarely in comparison to the rest of the period. Taking a constant CPR will give rise to misleading results since the prepayment in those years will be overestimated. In order to handle this shortcoming, Public Securities Association (PSA) model has been developed. This model combines Federal Housing Administration (FHA) experience with the CPR method.

2.3 *FHA Experience*

FHA is one of the federal agencies who provides insurance to mortgages of the qualified borrowers. FHA builds yearly a survivor table. This table consists of a row of thirty numbers that each represents the annual survivorship rates of FHA-insured mortgages. The prepayment rates are derived from this table. Although FHA experience is not often being used nowadays, this has been a widely used prepayment model. This model is based on historical prepaid mortgages and therefore is unable to project the future prepayments.

2.4 *PSA Model*

PSA model has been developed by combining FHA experience with CPR method. This method implies that the prepayment is zero at the initial month and each month increases with increments of 0.2% up to 30 months and then levels off till the end. Thus there is a linear increase from 0% to 6% until the thirtieth month and then it remains constant. This is called 100% PSA. Multiples of PSA are available employing different coefficients to the slope. For 150% PSA the CPR will be 0.3% in month one and increase with the increments of 0.3%. See the figure below for an illustration:

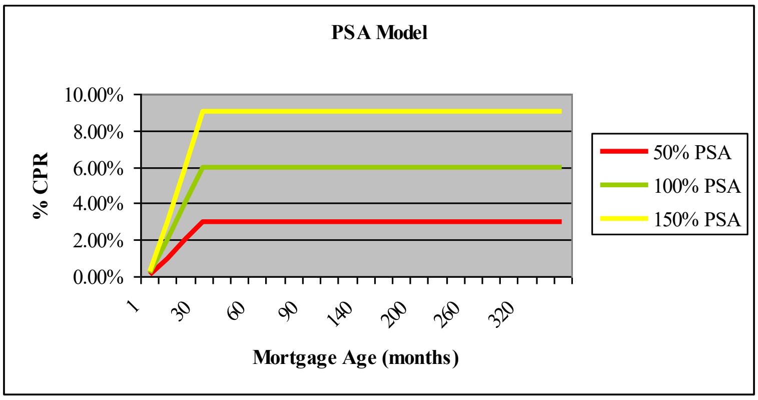

Figure 2-1

3 Cash Flow Structure of an MBS

Since we have worked out the prepayment phenomenon, one might expect to discuss the cash flow structure with and without the prepayment effect.

3.1 *Cash Flows without Prepayment*

We assume that the mortgage pool consists of fixed rate loans without prepayment of any mortgagor. Then the equal monthly instalments are:

$$MP = MB_0 \times \frac{[i(1+i)^n]}{[(1+i)^n - 1]}$$

3-1

where

MP is monthly mortgage payment (€)

MB_0 is original mortgage balance (€)

i is simple monthly interest rate (annual interest rate/12)

n is number of months.

The remaining principal balance at month t will then be:

$$MB_t = MB_0 \times \frac{(1+i)^N - (1+i)^t}{(1+i)^N - 1}$$

3-2

where

N is the maturity of the mortgage in months.

The initially scheduled interest payment at month t is:

$$I_t = MB_{t-1} \times i$$

3-3

The total cash flow at month t is the change of principal balance from t-1 to t and the interest rate excluding the service fee:

$$CF_t = (MB_{t-1} - MB_t) + \left(\frac{C}{C+S}\right) \times I_t$$

3-4

where

C is the coupon rate of the MBS and

S[4] is the servicing fee.

3.2 *Cash Flows with Prepayment*

Cash flows with prepayment option differ from those without prepayment which could be considered as a risk-free bond. We assume that there are in total *K* mortgagers in the pool and each one owns the same amount of mortgage loan. This implies that we could switch the concept of prepayment ratio measured in terms of money into the ratio measured in terms of number of remaining mortgagors. We also assume that there is no partial prepayment. In case of partial prepayment, each mortgagor will tend to prepay in different amounts caused by different factors mentioned earlier. This will add another dimension of complexity to the actual problem. We should then not only be interested in number of prepaid mortgagors but also the amount of each prepayment. However, if prepayment occurs, the mortgage will be fully prepaid. Denote the random variable $L_t =$ the number of mortgagors who prepay up to t.

Then the actual principal balance for month t is expressed in terms of L_t and K is

$$\overline{MB}_t = \frac{MB_t}{K} \times (K - L_t) = MB_t \times \left(1 - \frac{L_t}{K}\right),$$

3-5

and the actual interest at month t is

$$\bar{I}_t = \overline{MB}_{t-1} \times i = MB_{t-1} \times \left(1 - \frac{L_{t-1}}{K}\right) \times i = I_t \times \left(1 - \frac{L_{t-1}}{K}\right)$$

3-6

The total cash flow at month t is the change of the actual principal balance from t-1 to t and the interest rate excluding the service fee:

$$\overline{CF}_t = (\overline{MB}_{t-1} - \overline{MB}_t) + \left(\frac{C}{C+S}\right) \times \bar{I}_t$$

3-7

If we work the equation out:

[4] A percentage of each mortgage payment made by a borrower to a mortgage servicer as compensation for keeping a record of payments, collecting and making escrow payments, passing principal and interest payments along to the note holder, etc. Servicing fees generally range from 0.25-0.5% of the remaining principal balance of the mortgage each month (source: www.investopedia.com)

$$\overline{CF}_t = \left[MB_{t-1} \times \left(1 - \frac{L_{t-1}}{K}\right) - MB_t \times \left(1 - \frac{L_t}{K}\right)\right] + \left(\frac{C}{C+S}\right) \times I_t \times \left(1 - \frac{L_{t-1}}{K}\right)$$

$$= a_t \times \left(1 - \frac{L_t}{K}\right) + b_t \times \left(1 - \frac{L_{t-1}}{K}\right)$$

3-8

where

$$a_t = -MB_t$$

$$b_t = MB_{t-1} + \left(\frac{C}{C+S}\right) \times I_t$$

4 The models

In this chapter we would like to present various well-known models. We will discuss the valuation model of Stanton (1995) in which the prepayment model is based on rational decisions by mortgage holders; the model of Schwartz and Torous (1989) who develop their model using maximum likelihood techniques to estimate prepayment function; the model of Dunn and McConnell (1981) based on suboptimal prepayment decision and the 3-factor valuation model of Kariya, Ushiyama and Pliska which is an extended framework of the Kariya&Kobayashi valuation model (2000). Of course, we can discuss more models but since those mentioned above have been a starting point for investors and financial engineers, we will restrict us to these basic frameworks. Finally, in the last chapter we will compare those models on their abilities.

4.1 *A 3-factor Valuation Model for Mortgage-Backed Securities*

The one-factor MBS-pricing model of Kariya and Kobayashi (hereinafter KK (2000)) is extended to a 3-factor model through separating the mortgage rate process from the short-term interest rate process which are highly correlated with each other and including the equity factor which is related to rising housing prices. This model assumes that the prepayment behaviour is heterogeneous and embeds this into the valuation while some other models assume that the prepayments are homogeneous and the value of the prepayment option is regarded as a gross or lump-sum value and the value of MBS will then be decomposed into a prepayment option part and the value of a riskless bond.

In KK (2000) model, by a general no-arbitrage pricing theory for a discrete time framework, the no-arbitrage value at time *m* of the MBS with maturity *N* using the cash flow structure in section 3.2 is given by

$$V(m,N) = \sum_{n=m+1}^{N} CF(m,n)$$

4-1

where

$$\begin{aligned} CF(m,n) &:= E_m^*[\Delta(m,n)CF_n] \\ &= E_m^*\left[\Delta(m,n)\left\{a_n\left(1-\frac{L_n}{K}\right)+b_n\left(1-\frac{L_{n-1}}{K}\right)\right\}\right] \end{aligned}$$

4-2

with

$$\Delta(m,n) := \exp\left(-\sum_{j=m}^{n-1} r_j h\right)$$ (h=1/12)

4-3

$\{r_j\}$ is the short-term riskless interest rate process and $E_m^*[\cdot]$ is the conditional expectation at *m* under a risk neutral measure for the interest rate process $\{r_j\}$, the mortgage rate process $\{R_j\}$ and the housing price level $\{P_j\}$. Here $\Delta(m,n)$ randomly discounts a cash flow at n to value at m.

In KK (2000) model, prepayment in a mortgage pool takes place at *n* when the difference between the initial mortgage rate R_0 and the current rate R_n first exceeds his incentive threshold for the first time. If we assume that each mortgager has their own threshold d_k, the condition for the prepayment can be formulated as follows:

$$R_0 - R_n \geq d_k^{(1)}$$

4-4

KK (2000) assumed that the mortgage rate process $\{R_j\}$ and the interest rate process $\{r_j\}$ are highly correlated with each other, so the $\{r_j\}$ process determines the prepayment due to refinancing via 4.4 and the discount factor in 4.3. The threshold d_k, which in principle depends on time and different variables, is assumed constant over time for sake of simplicity.

Another factor that affects the prepayment behaviour is the equity factor according to Kariya, Ushiyama and Pliska (2002). They call P_n the housing price level at time *n* and the k-th mortgagor sells his house if the difference of current log-price and the initial log-price exceeds his threshold which is conditioned as follows:

$$\log P_n - \log P_0 \geq d_k^{(2)}$$

4-5

This threshold is also taken constant over time for simplicity.

After giving the conditions, they specify those three factors affecting the prepayment as follows:

- $\Delta r_n = \theta_0^{(0)}(\theta_1^{(0)} - r_{n-1})h + \theta_2^{(0)}\sqrt{h\varepsilon_n^{(0)}}$ (discrete time Vasicek model)
  4-6
- $\Delta R_n = \theta_0^{(1)}(\theta_1^{(1)} - R_{n-1})h + \theta_2^{(1)}\sqrt{h\varepsilon_n^{(1)}}$ (discrete time Vasicek model)
  4-7

where $\Delta r_n = r_n - r_{n-1}$, $\Delta R_n = R_n - R_{n-1}$, h=1/12 and the θ_i^j's are various scalar parameters.

- $P_n = P_{n-1}\exp\left(\mu_{n-1}h + \sigma\sqrt{h\varepsilon_n^{(2)}}\right)$
  4-8

where $\mu_{n-1} = \phi\mu_{n-2} + (1-\phi)\log\left(\frac{P_{n-1}}{P_{n-2}}\right)$, the volatility (σ) is assumed to be constant and the parameter ϕ satisfies $0 \le \phi \le 1$. The value $1-\phi$ is explained as the proportion of a recent change in price brought into a change in the drift. The greater the $1-\phi$ is, the more volatile the drift is, though it depends on the volatility σ . $\varepsilon = (\varepsilon_n^{(0)}, \varepsilon_n^{(0)}, \varepsilon_n^{(2)})$ are assumed to be i.i.d. as 3-dimensional normal random variables with $\mu = 0$ and covariance matrix **Λ,** where

$$\Lambda := \begin{pmatrix} 1 & \rho_{01} & \rho_{02} \\ \rho_{10} & 1 & \rho_{12} \\ \rho_{20} & \rho_{21} & 1 \end{pmatrix}$$

They finalize the model using the following assumption since the distribution of the thresholds plays an important role in determining the prepayment and as a result the price of an MBS: The K pairs of random variables $\{d_k^{(1)}, d_k^{(2)} : k = 1,...K\}$ are i.i.d. with 2-dimensional normal distribution *N(**μ**,**Σ**)*, where

$$\mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} (\sigma^{(1)})^2 & \sigma^{(1)}\sigma^{(2)}\delta \\ \sigma^{(1)}\sigma^{(2)}\delta & (\sigma^{(2)})^2 \end{pmatrix}$$

4-9

It is possible to estimate the expectations stated in 4-2 and worked out above, using Monte Carlo simulation in order to calculate the theoretical value of the MBS.

4.2 *Valuation Model with Maximum Likelihood Techniques*

Schwartz and Torous (1989) have developed an empirical valuation using maximum likelihood techniques. They have estimated the prepayment function by a proportional-hazards model using the historical information from the past GNMA[5] experience. More specifically, the influence of various explanatory variables or covariates on the mortgagor's prepayment decision was estimated. They have also explicitly modeled the effects of seasoning. They have finally come

[5] The Great majority of mortgage–backed pass-throughs have been issued by three agencies. The Federal National Mortgage Association (FNMA, or "Fannie Mae"), the oldest of these agencies, was established by the federal government in 1938 to help solve some of the housing finance problems brought on by the Great Depression. In 1968, Congress divided the original FNMA into two organizations: the current FNMA and the Government National Mortgage Association (GNMA, or "Ginnie Mae"). GNMA remains a government agency within the Department of Housing and Urban Development (HUD), helping to finance government-assisted housing programs (source: The Handbook of Fixed Income Securities Fourth Edition–Fabozzi 1995)

with a second order partial differential equation and used Monte Carlo simulation methods to solve the equation that is subject to boundary and terminal conditions which characterize the particular mortgage-backed security.

They have modeled the prepayment function by a proportional-hazards model as follows:

$$\pi(t; \underline{v}, \underline{\theta}) = \pi_0(t; \gamma, p) \exp(\underline{\beta} \underline{v}),$$

4-10

where the base-line hazard function $\pi_0(t; \gamma, p)$ is given by the log-logistic hazard function

$$\pi_0(t; \gamma, p) = \frac{\gamma p (\gamma t)^{p-1}}{1 + (\gamma t)^p}$$

4-11

In equation 4-10, $\underline{v}$ is the vector of explanatory variables which will be formulated below and $\underline{\beta}$ is the vector of coefficients. Equation 4-11 measures the probability of prepayment under homogeneous conditions ($\underline{\beta} = \underline{0}$).

Schwartz and Torous employ the refinancing behavior of mortgagers into the prepayment function using the following covariate:

$$v_1(t) = c - l(t - s), \ s \geq 0$$

4-12

where *c* stands for contract rate and *l* for long-term Treasury rate that is applied at time *t-s*. This covariate represents the relationship between rates at which the mortgage may be refinanced and the contract rate on the mortgage. More specifically, if available refinancing rate is less than the contract rate, there exists an incentive to prepay. The larger $v_1(t)$ is, the greater is this incentive to prepay.

Another covariate that accelerates the prepayment when the refinancing rates are sufficiently lower than the contract rate is:

$$v_2(t) = (c - l(t - s))^3, \ s \geq 0$$

4-13

To be able to represent the burnout effect, the following covariate has been used:

$$v_3(t) = \ln\left(\frac{AO_t}{AO_t^*}\right)$$

4-14

where AO_t represents the dollar amount of the pool outstanding at time t while AO_t^* is the outstanding in the absence of prepaid portion.

The last covariate employs the seasonality into the prepayment function which is represented as follows:

$$v_4(t) = \begin{cases} +1 & if \quad t = May - Aug. \\ 0 & if \quad t = Sept. - April \end{cases}$$

4-15

Given the prepayment function with four explanatory variables and the past GNMA prepayment experience, a statistical maximum likelihood estimation (MLE) of those variables is possible since those variables influences the mortgagor's prepayment decision. The parameters that will be estimated are the prepayment function's parameter values represented within $\underline{\theta} = (\gamma, p, \beta_1, \beta_2, \beta_3, \beta_4)$ with the inclusion of covariates represented within $\underline{v}(t) = (v_1(t), v_2(t), v_3(t), v_4(t))$. We will restrict us to the prepayment function modeled above and the valuation formula of an MBS which will be discussed below and not go further into detail how those values are estimated using MLE to have this paper to the point.

In order to value an MBS, Schwartz and Torous first presented the payout rate and the principal outstanding of a default-free fixed-rate fully amortizing mortgage. In a continuous-time valuation model the payout rate of a mortgage with a contract rate *c* and maturing at *T* years is:

$$A = \frac{cP(0)}{1 - \exp(-cT)}$$

4-16

and the principal outstanding at time t is:

$$P(t) = \frac{A}{c}(1 - \exp(-c(T - t)))$$

4-17

where P(0) is the principal of mortgage at origination.

They have made couple of assumptions before coming with the partial differential equation. Those assumptions are:

- Risk-free interest rate *r* (short-term rate) and default-free bond yield *l* (long-term rate) are taken as term structure of interest rates
- Dynamics of *r* and *l* (following Brennan and Schwartz (1982)) are assumed as

$$dr = (a_1 + b_1(l - r))dt + \sigma_1 r dz_1$$

4-18

$$dl = (a_2 + b_2 l + c_2 r)dt + \sigma_2 l dz_2$$

4-19

where z_1 and z_2 are standardized Wiener processes and their increments are correlated with correlation coefficient ρ:

$$dz_1 dz_2 = \rho dt$$

4-20

- Prepayment rate is assumed to be

$$\pi = \pi(l, x, y, t; c)$$

4-21

where the time-varying *x(t)* denotes the history of past interest rates and *y(t)* the burnout effect. According to Ramaswamy and Sundaresan (1986) the state variable *x(t)* is defined by

$$x(t) = \alpha \int_{\infty}^{0} \exp(-as) l(t-s) ds \text{ , } \alpha > 0$$

4-22

This exponential average of historical bond yields captures the effects of past refinancing rates on current prepayment decisions. Its stochastic differential equation is given by

$$dx = \alpha(l - x)dt$$

4-23

The burnout effect y(t) captures the heterogeneity in mortgagors and its stochastic differential equation is given by

$$dy = -y(\pi + AP^{-1}(t) - c)dt$$

4-24

Given the above assumptions the value of an MBS is given by

$$B = B(r, l, x, y, t)$$

4-25

The PDE that the value of an MBS must satisfy is given by

$$\frac{1}{2} r^2 \sigma_1^2 B_{rr} + rl\rho\sigma_1\sigma_2 B_{rl} + (a_1 + b_1(l - r) - \lambda_1\sigma_1 r)B_r + l(\sigma_2^2 + l - r)B_l$$

$$+ \alpha(l - x)B_x - y(\pi + AP^{-1}(t) - c)B_y + B_t - (r + \pi)B + \pi P(t) + A = 0$$

4-26

where λ1 denotes the market price of short-term interest rate risk.

Given that the mortgage is fully amortizing, the following terminal boundary condition must be satisfied as well:

$$B(r,l,x,0,T)=0$$

4-27

which deduces that there will be no prepayment at the maturity.

The coefficients of the PDE in 4-26 depend upon the parameters of the interest rate processes. To be able to implement the MBS valuation in light of the previously estimated prepayment function, these parameters should be estimated using MLE as well. Below you find a table with the estimation results corresponding to a specific time period.

Maximum-Likelihood Estimates of Interest Rate Process Parameters

We provide maximum-likelihood estimates, with standard errors in parentheses, of the parameters of the interest rate processes $dr = (a_1+b_1(l-r))dt + \sigma_1 r dz_1$ and $dl = (a_2+b_2 l+c_2 r)dt + \sigma_2 l dz_2$ with $(dz_1)(dz_2) = \rho dt$.

a_1	b_1	a_2	b_2	c_2	σ_1	σ_2	ρ
−0.800	0.0382	−0.0033	−0.0007	0.0008	0.0262	0.0173	0.3732
(0.0359)	(0.0174)	(0.0063)	(0.0019)	(0.0016)			

Table 4-1

The estimated parameters are based on data which is collected from 1982 until 1987. Data consists of two types of information, namely short-term Certificates of Deposit (CD) and long-term U.S. Treasury bonds. The estimated parameters a_1 and b_1 from the drift of the short-term interest rate process are statistically significant while the estimated parameters a_2, b_2 and c_2 from the drift of the long-term interest rate process are insignificant. This justifies the theory that under no-arbitrage arguments, long term interest rates have no drift and follow a random walk.

Finally, Monte Carlo simulation methods are employed to solve the PDE in equation 4-26, subject to terminal boundary condition in equation 4-27. Since the insignificance of the coefficients of the long-term interest rate process holds, Monte Carlo simulation methods require that r and l are generated by the following risk-adjusted processes:

$$dr=(a_1+b_1(l-r)-\lambda_1\sigma_1 r)dt+\sigma_1 r dz_1$$

4-28

$$dl = l(\sigma_2^2 + l - r)dt + \sigma_2 l dz_2$$

4-29

Further to this, the monthly cash flows of the MBS with the prepayment probability incorporated (which is derived from the prepayment function) will be discounted at the randomly generated normal variables corresponding to *r* and *l* (also correlated with *ρ*). The total present value of these cash flows represents the value of an MBS.

4.3 *Valuation Model with Suboptimal Prepayment Decision*

Dunn and McConnell (1981) introduce a PDE approach to value the GNMA mortgage-backed securities using suboptimal prepayment decision of mortgagors.

GNMA mortgage-backed pass-through securities are issued generally by mortgage bankers, who are approved by the FHA. Each month the issuer of this security must "pass through" the scheduled interest and principal payments on the underlying mortgage to the holder of the security whether or not the issuer has actually collected those payments from the mortgagors. If the mortgage bankers default to make the payments, GNMA undertakes for timely payment of principal and interest. Those securities are considered to be risk-free instruments.

Dunn and McConnell firstly introduce the "generic model" for pricing interest contingent securities developed by Brennan and Schwartz (1977) and Cox, Ingersoll and Ross (1978) and then embed the suboptimal prepayment option into the model. The generic model is derived from the following assumptions:

1. The value of default-free fixed interest rate security, $V(r(t), \tau)$, is a function only of the current value of the instantaneous risk-free rate, *r(t),* and its term to maturity τ.
   This assumption means that the current risk-free interest rate completely summarizes all information which is relevant for the valuation of fixed-rate securities, e.g. bonds.
2. The interest rate follows a continuous stationary Markov process given by the stochastic differential equation

   $$dr = \mu(r)dt + \sigma(r)dz$$

   4-30

   where

   $\mu(r) \equiv k(m - r)$ *k, m>0,*

   $\sigma(r) \equiv \sigma\sqrt{r}$, σ constant, *dz* is a Wiener process with *E(dz)=0* and $dz^2=dt$ with probability 1. The function *μ(r)* is the drift of the process; *k* is the speed of the adjustment parameter, *m* is the steady state mean of the process and the function $\sigma^2(r)$ is the variance.

3. The risk adjustment term, $p(r)\sigma\sqrt{r}$ is proportional to the spot interest rate, i.e.

   $$p(r)\sigma\sqrt{r} = qr$$

   4-31

   where *q* is the proportionality factor and *p(r)*, the price of interest rate risk, equals the equilibrium expected instantaneous return in excess of the riskless return per unit of risk for securities which satisfy the first assumption.
4. Individuals have risk preferences consistent with equation 4-31 and agree on the specification of equation 4-30.
5. The capital market is competitive; trading takes place continuously which eliminates arbitrage possibility.
6. The cash flows $C(\tau)$ from any security are paid continuously.

Assumptions 1 to 5 lead to the model of term structure of interest rates introduced by Cox, Ingersoll and Ross[6]. This term structure of interest rates provides the foundation for the GNMA pricing model according to Dunn and McConnell.

Using the assumptions above, Dunn and McConnell come with the following generic model:

$$\frac{1}{2}\sigma^2(r)V_{rr}{}^{7} + \left[\mu(r) - p(r)\sigma(r)\right]V_r - V_\tau - rV + C(\tau) = 0$$

4-32

with the initial condition[8]

$$V(r,0) = F(0)$$

4-33

and with the boundary conditions

[6] Also known as CIR Model

[7] $V_{rr} := \dfrac{\partial^2 V(r(t),\tau)}{\partial \tau^2}$

[8] At maturity, *τ=0*, the value of a default-free bond must equal its face value or remaining principal balance *F(0)*

$\lim_{r \to \infty} V(r,\tau) = 0$ [9]

4-34

$V(r,\tau) \leq F(\tau)$ [10]

4-35

Dunn and McConnell introduce suboptimal prepayments and embed this into the generic model. The motivation is that sometimes mortgagors call their loans at times other than those that would be dictated by the optimal call policy. There are also cases that occur when *r* is above r_c as suboptimal prepayments. The prepayments are then suboptimal only in the sense that the amount of the prepayment exceeds the market value of the debt. The additional assumptions for this characteristic are:

7. Prepayments which occur when the value of a GNMA security is less than its remaining principal balance follow a Poisson-driven process. The Poisson random variable, *y*, is equal to zero until the loan is called suboptimal. If *y* jumps to one, there is a suboptimal prepayment and the security ceases to exit. The Poisson process *dy* is given by

   $$dy = \begin{cases} 0 & if \quad \text{a suboptimal prepayment does not occur} \\ 1 & if \quad \text{a suboptimal prepayment occurs} \end{cases}$$

   where

   $E(dy) = \lambda(r,\tau)dt$

   4-36

   and $\lambda(r,\tau)dt$ is the probability per unit of time of a suboptimal prepayment at a time to maturity *τ* and interest rate *r*.

8. prepayments which occur when the value of a GNMA security is less than its remaining principal balance are uncorrelated with all relevant market factors and are, therefore, purely non systematic

Adding those two new assumptions and substituting for *μ(r)* and *σ(r)* from 4-30 and for *p(r)* from 4-31, we obtain

---

[9] The value of an interest contingent security goes to zero as the interest rate approaches infinity.

[10] For each *τ* there is some level of the risk free interest rate, say $r_c(\tau)$, for which $V[r_c(\tau), \tau]=F(\tau)$ and the call option will be exercised. Risk-free interest rates below $r_c(\tau)$ are not relevant for pricing callable bonds. The effect of the optimal call policy is to preclude the market value of a bond from exceeding its remaining principal balance; therefore the boundary condition

$$\frac{1}{2}\sigma^2 r V_{rr} + [km - (k+q)r]V_r - V_\tau - rV + C(\tau) + \lambda(r,t)[F(\tau) - V] = 0$$

4-37

Equation 4-37 together with the initial condition 4-33 and the boundary conditions 4-34 and 4-35 can be solved for the value of a GNMA MBS.

4.4 *Valuation Model Based on Rational Decisions*

This valuation model of Stanton (1995) which is assumed as a breakthrough among structural models is actually an extension of the rational prepayment models. Stanton first lays out the model through describing prepayment decision process of a single rational mortgagor and then works the model out in detail.

He introduces first the value of a mortgage liability as a bond price minus the option value of the prepayment. The bond price here is nothing else than the present value of the remaining cash flow streams on the mortgage. The call option on bond *B* at *t*, B_t, has an exercise price of

$$F_t(1 + X_i)$$

4-38

where

F_t is the remaining balance at t and

X_i is the transaction cost of mortgagor i associated with prepayment.

The value of a mortgage liability, M_t^l, is given by

$M_t^l = B_t - V_t^l$ (*l* for liability)

4-39

where V_t^l is the value of prepayment option to the mortgagor. Since B_t does not depend on the mortgagors prepayment decision, minimizing the liability value is equivalent to maximizing the option value.

Further, Stanton distinguishes between endogenous and exogenous prepayment reasons. An endogenous reason is that the mortgagor prepays his loan if the refinancing rates are sufficiently below the contract rate. Exogenous reasons are such as divorce, moving or sale of the house. The likelihood of an endogenous prepayment is described by a hazard function *ρ* and the likelihood of an exogenous prepayment is described by another hazard function *λ.*

The value of a mortgage-backed security whose cash flows are determined by the prepayment behavior of the mortgagor is

$M_t^a = B_t - V_t^a$ (*a* for asset)

There is a difference between asset and liability values because of the transaction costs associated with prepayment. These costs paid by the mortgager are not received by the investor of the MBS.

For a given coupon rate and transaction cost X_i, there is a critical interest rate r_i^* such that if $r_t \le r_i^*$ the mortgagor will optimally prepay. Equivalently, for a given coupon rate and interest rate r_t, there is a critical transaction cost X_t^* such that if $X_i \le X_t^*$ the mortgage holder will optimally prepay. This exercise strategy defines a hazard function that holds for a single mortgagor. The hazard rate governing prepayment is given by

$$\begin{cases} \lambda & if \quad r_t > r_t^* \quad (equivalently, \quad X_i > X_t^*) \\ \lambda + \rho & if \quad r_t \le r_t^* \quad (equivalently, \quad X_i \le X_t^*) \end{cases}$$

4-40

Probability of total prepayment is given by

$$\text{P(prepayment)=}\begin{cases} P_e = 1 - \exp\left(\frac{-\lambda}{12}\right) \\ P_r = 1 - \exp\left(\frac{-(\lambda + \rho)}{12}\right) \end{cases}$$

4-41

where P_e denotes for the probability of the prepayment this month if only exogenous prepayment will occur[11] and

P_r denotes for the probability of prepayment this month if it is optimal to prepay.

The assumption that Stanton makes about the interest rate process is that he employs the one-factor CIR model such as other authors. In this model r_t satisfies the following SDE:

$$dr_t = \kappa(\mu - r_t)dt + \sigma\sqrt{r_t}dz_t$$

4-42

On average the interest rate *r* converges towards to *μ* and the parameter *κ* governs the rate of this convergence process. $\sigma\sqrt{r_t}$ is the volatility of interest rates. One further parameter q is needed to value of the mortgage since it summarizes risk preferences of the representative individual mortgagor.

[11] It is not optimal to prepay for endogenous reasons such as interest rate.

Given the CIR model, one might formulate the value of the mortgage and the optimal exercise strategy that satisfies the PDE. $V(r_t,t)$, the value of an interest rate contingent claim paying coupons at some rate $C(r_t,t)$ must satisfy

$$\frac{1}{2}\sigma^2 rV_{rr} + [\kappa\mu - (\kappa + q)r]V_r + V_t - rV + C = 0$$

4-43

Stanton uses the Crank-Nicholson algorithm to get a price for an MBS. This algorithm works backward to solve the PDE. This gives the value of an MBS conditional on the prepayment option remaining unexercised, $M_u^a(y,t)$. $M^a(y,t)$ is a weighted average of $M_u^a(y,t)$ and $F(t)(1+X)$ is the probability that the mortgage will be prepaid in month t.

To summarize Stanton's comprehensive valuation model:

- Discretize time [0,T] into N steps, $\delta t = \frac{T}{N}$
- Each step unprepaid MBS value M_u^a is obtained by solving the PDE in equation 4-43
- MBS value at each step is given by

$$M^a(y,t) = \begin{cases} (1-P_e)M_u^a(y,t) + P_eF(t) & if \quad M_u^a \le F(t)(1+X), \\ (1-P_r)M_u^a(y,t) + P_rF(t) & otherwise \end{cases}$$ [12]

4-44

- Repeat these steps until t_0=0

[12] y transforms natural boundaries of interest rate grid, $[0,\infty]$, onto the finite range $[0,1]$

5 The Conclusion

In this section we discuss the models that are worked out in the previous section. Of the four models we can easily eliminate two models since they have major shortcomings. However, those two have surely been enlightenment for further research.

Schwartz and Torous have empirically modeled the prepayment as a function of some set of explanatory variables using the past prepayment rates. Their aim was to fit the shape of observed prepayment data, unrestricted by many theoretical considerations. It is not clear how this model will perform in a different economic environment. If the interest rate process or mortgage contract terms would change, mortgage prepayment behavior would also change. Purely empirical models, such as this model can make no predictions about the magnitude of this change.

Dunn and McConnell have determined the prices and prepayment behavior together, both depending on the assumed interest rate model. This model links prepayment and valuation within a single framework, allowing it to address what would happen in the event of structural shift in the economy. However, their model implies arbitrage bounds[13] on mortgage backed securities that are often violated in practice. Major shortcoming is that mortgagors may not prepay even when it is optimal to do so.

Kariya, Ushiyama and Pliska (KUP) have extended the framework of Kariya and Kobayashi (2000) through making a distinction between the short-term rates and the mortgage rates and adding the equity factor which is related to rising housing prices. This distinction is very important and also holds for all other models because a decrease in the mortgage rate will tend to lower the MBS value due to refinancing, whereas this decrease will also tend to increase the MBS value since the discount factors will increase. Besides this, their framework directly embeds prepayment behavior into the valuation of an MBS such as that of Dunn and McConnell. In their prepayment behavior, exogenous reasons are not included but the model is such that they can easily be embedded. Further to that, the thresholds in determining prepayments are randomly distributed per mortgagor but taken constant over time while in reality these should depend upon time and other factors. This shortcoming is eliminated in Stanton's model through introducing refinancing costs that vary over time.

[13] Theoretical MBS value is bounded from above, see equation 4-35

Like Dunn and McConnell and KUP, the model of Stanton embeds prepayment behavior directly into the valuation and therefore changes in interest rates will directly effect the cash flow structures and as a result the value of an MBS. This model extends the model of Dunn and McConnell in several ways:

- It explicitly models and estimates the heterogeneity in the transaction costs faced by mortgagors,
- Mortgagors make prepayment decisions in discrete time intervals, rather than continuously.

These two features produce endogenously the burnout effect noted in empirical studies without the need to specify an ad hoc exogenous burnout factor.

- The model gives rise to a simple reduced form representation for prepayment.

In light of the above analysis, we find that Stanton's model is more comprehensive in comparison to others. But the model of KUP answers most of the shortcomings and easy to implement using Monte Carlo simulation while Stanton's model is only a numeric algorithm. However, these two can be considered as reasonable ones.

Reference List

Dunn, K. B., and J. J. McConnell [1981b], “Valuation of GNMA Mortgage-Backed Securities”, Journal of Finance 36: 599-617.

Fabozzi, F.J. [1995], “The Handbook of Mortgage Backed Securities Fourth Edition”,

Schwartz, E. S., and Torous, W. N., [1989], “Prepayment and the Valuation of Mortgage-Backed Securities”, Journal of Finance 44: 375-392.

Stanton, R. H. [1995], “Rational Prepayment and the Valuation of Mortgage-Backed Securities”, Review of Financial Studies 8: 677-708.

Rom-Poulsen, N. [2007], “Semi-Analytical MBS Pricing”, J Real Estate Finan Econ (2007) 34:463–498

Kariya, T. and Kobayashi, M., [2000], “Pricing Mortgage-Backed Securities (MBS)- A Model Describing the Burnout Effect”, Asia-Pacific Financial Markets 7, 189 - 204

Kariya, T., Ushiyama, F. and Pliska, S., [2002], “A 3-factor Valuation Model for Mortgage-Backed Securities (MBS)”, working paper.

Schwartz, E. S., and Torous, W. N., [1992], “Prepayment, Default, and the Valuation of Pass-through Securities”, Journal of Business 65: 221-240.

Nakamura, N., [2001], “Valuation of mortgage-backed securities based upon a structural approach”, Working paper.

Boudhouk, J., M. Richardson, R. Stanton, and R.Whitelaw, [1997], “Pricing mortgage backed securities in a multifactor interest rate environment: a multivariate density estimation approach”, Review of financial studies, 10, 405 - 446.

Collin-Dufresne, P., and Harding, J. P. [1999]. “A closed form formula for valuing mortgages”. The Journal of Real Estate Finance and Economics, 19(2), 133–46.

Appendix A: Derivation of Monthly Mortgage Payment

Amount owed at month 0: MB_0

Amount owed at month 1: $(1 + i)\ MB_0 - MP$

Amount owed at month 2: $(1 + i)((1 + i)\ MB_0 - MP) - MP = (1 + i)^2\ MB_0 - (1 + (1 + i))\ MP$

Amount owed at month 3: $(1 + i)((1 + i)((1 + i)\ MB_0 - MP) - MP) - MP = (1 + i)^3\ MB_0 - (1 + (1 + i) + (1 + i)^2)\ MP$

. . .

. . .

Amount owed at month *N*: $(1 + i)^N\ MB_0 - (1 + (1 + i) + (1 + i)^2 + (1 + i)^3 + \ldots + (1 + i)^{N-1})\ MP$

The polynomial $p_N(x)=1 + x + x^2 + \ldots + x^{N-1}$ with $x=(1+i)$ has a simple closed-form expression obtained from observing that $xp_N(x) - p_N(x) = x^N - 1$ because all but the first and last terms in this difference cancel each other out. Therefore, solving for $p_N(x)$ yields the much simpler closed-form expression which can be formulated as:

$$p_n(x) = 1 + x + x^2 + \ldots + x^{N-1} = \frac{x^N - 1}{x - 1}$$

Applying this fact to the amount owed at *N*th month:

Amount owed at month *N*

$$= (1+i)^N MB_0 - p_N(1+i)MP$$

$$= (1+i)^N MB_0 - \frac{(1+i)^N - 1}{1+i-1} MP$$

Since the amount owed at month *N* must be zero because the mortgagor agrees to fully pay off, the monthly mortgage payment MP can be obtained by:

$$MP = MB_0 \frac{i(1+i)^N}{(1+i)^N - 1}$$

(Source: Wikipedia - http://en.wikipedia.org/wiki/Mortgage_Calculator)