American Options; an American delayed- Exercise model and the free boundary

Business Analytics Paper

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Abstract

In this paper, we will shed light on the new and so called Ameripean Delayed-Exercise Model. This model uses the first moment that the option value crosse the payoff function, as the beginning of a waiting time period before the exercise, which can be seen as a free-boundary problem. Basically this model uses the assumptions of the well- known American option, however it is more complicated. Different numerical aspects of it will be described in this paper.

Keywords: delayed exercise, option-pricing theory, ParAsian options, American options, European options, Black–Scholes equation, Free-Boundary problem

Preface

Writing a research paper is a compulsory part of the Business Analytics master at the VU University. The essential purpose of this paper is providing a numerical method to solve and detect the free boundary for pricing the American put option.

I would like to take this opportunity to thank my supervisor for his patience and support.

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1.Introduction

In this study, we will address a new class of option models, namely "The Ameripean delayed-exercise model", which has features of American and ParAsian[1] options. As it is explained in paper [1], this model uses the first time that the option value crosses the payoff function as a trigger to start a new period for delayed exercise. Delaying of the exercise was a suboptimal outcome, however we have to be careful when choosing the right moment to start it. So it is important to know why would we need to model such an option. Paper [1] explained two main reasons for studying such options. First, the early exercise feature is not always safe for the holder. Second, the option can give a fluid link between the European option and the American option.

The option holders have different behaviors, some of them are eager and do not like to take any risk; they are “risk averse” for choosing the time to start exercising the option, while some of them can take the risks of delaying exercise (for more details see [4]). Therefore, the behavior is important in identifying the pricing model which should be used. This was one of the reasons that motivated the authors of paper[1] to study this new model. In [5], Gauthier has shown that ParAsian options are more effective than classical hitting times. The ParAsian options are especially better for explaining the optimal decision to invest in a project when there is time delay between the implementation and the decision. However, using only an arbitrary barrier in the decision variable combined with a ParAsian feature to model the delay before the implementation, does not ensure the optimality in the timing of the decision to invest. The study described in [6] suggests a mortgage valuation framework, as a model of the prepayment behavior of the borrowers where the barrier is not like the value of the underlying state an arbitrarily set. In fact paper [8] has explained a mortgage valuation framework modeling the prepayment behavior of borrowers, on the basis of that the barrier is not arbitrarily set as a value of the underlying but is implied with the same constraint value process, comparable to that when the decision is made by an American call option. This would not only keep the optimal timing of the decision, but also allows the delay in the enforcement of the transaction. This makes keeping the optimizing constraint within this framework a very interesting choice. In the remainder of this paper, we will analyze this option in detail using the cumulative (ParAsian) version. Actually in the model explained in paper[1], instead of the holder exercising when it is optimal to do so (i.e. when $\bar{\tau} \to 0$ in terminology of the paper[1], to be defined later), the holder may choose to wait and hold the option. It is clear, if the holder waits long enough, the option reduces to a vanilla European option. That is why this model provides a link between the European option and American option.

This paper proceeds further as follows: In section 2 some important definitions and details about the new option model are provided, the free boundary problem and in particular the American Put option, and a perpetual case will be discussed. Section 3 contains an explicit numerical method. The results for the Ameripean put option are given in section 4. Computation of the free boundary of an American put option will given in section 5. Finally, the conclusion and summary will be provided in section 6.

[1] This class of option involves the underlying having to spend a prescribed amount of *cumulative* time above/below a prescribed barrier before exercise. See Haber, Sch¨onbucher, and Wilmott [3].

2. The "Ameripean" Delayed-Exercise Model

Before discussing the details of this new model, we will start with an overview of some of the exotic and path-dependent options. In general, a path-dependent option is an option with a payoff at exercise or expiry that depends on the past history of the underlying asset price in addition to its spot price at expiry. One of the path dependent options we are dealing with in this work is the American option[2]. Because the American option has usually a finite probability of being exercised before expiry, its payoff value will change as the underlying asset's value change. Another class of exotic option is Barrier options [2].

2.1 Barrier Options

A Barrier option is a path dependent option that loses its right to exercise if the asset value crosses a certain value (an out barrier), or gets its right to exercise only if the asset value crosses a certain value (an in barrier). Examples of common barrier options are a down-and-out option and an up-and-out option. We can define those two options briefly as follows respectively:
A down-and-out option is an option that loses its right to exercise if the asset price falls below or to some given barrier(down-or-out).
An Up-and-out option expires worthless when the barrier is reached from below before expiry.
Barrier options can have one or two boundaries, where both single and the two boundaries barrier option can be characterized in two categories, knock-in and knock-out. A knock-in option exists when the asset price reaches a certain boundary, and a knock-out option exists when the asset price reaches the barrier.

The "Ameripean" Delayed-Exercise Model, analyzes American and ParAsian advantages.
As we know, one of the advantages of dealing with American options, is that the holder of the option can exercise at any time before and until the expiration time. This means that by using this option model, the holder can exercise till the relaxed time[2].

First we will briefly explain the option pricing and value for the American put option in general. Furthermore, important model assumptions will be highlighted.
The value of an American put option is given by:

$$P_A(S,t) \geq \max(E-S,0) \tag{2.1}$$

The holder of the American put option will want to exercise when the value of the option is at or below the payoff function. So the payoff function here can be seen as a barrier to the option value process, where the option expires. But this is not the only barrier, there is a corresponding barrier for the asset price, where the optimal exercise boundary is $S_f(t)$,which is the optimal price for the holder to exercise.

In the following paragraph we are going to explain the American put options as free boundary problems in general, and see how paper [1] have explains this for the new class model.

[2] Relaxed time: it start at the moment that the option value crossed the payoff function.

2.2 American put option as a free boundary option

Suppose that the problem takes place on the area: $[0, \infty) \times [0, T]$. Thus $\forall t \in [0, T]$ if we want to separate the S axis into two intervals. The boundary between the subintervals will be given by a function $S_f(t)$. Suitable boundary conditions will hold on each of the subintervals and on the boundary between them. Since the location of that boundary is unknown in advance, we have then what is called a free boundary problem. As we have seen in [2], this free boundary problem can be explained as a differential inequality problem, as follows:

First region: $0 \leq S < S_f(t)$

In this case early exercise for this value of S is optimal;

$P = E - S$ and

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial^2 S} + rS\frac{\partial P}{\partial S} - rP < 0$$

this means that the holder of the American Put option has to exercise.

Second region: $S_f(t) < S < \infty$

For this S the early exercising is not optimal, where;

$P > E - S$ and

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial^2 S} + rS\frac{\partial P}{\partial S} - rP = 0$$

As explained in book [2], if S increased from $S_f(t)$ then the value of the put option $P(S, t)$ drops below the payoff $\max(E - S, 0)$, which contradicts with the inequality above.

On the "Free boundary": $S_f(t) = S$

In this case the value of the Put option and its slope(delta) have to be continuous:

$P(S_f(t), t) = \max(E - S_f(t), 0)$ and $\frac{\partial P}{\partial S}(S_f(t), t) = -1$

In fact, these two equations determine the location of the free-boundary $S_f(t)$ (see[2], Chapter 7).

In paper [1], the payoff function has been regarded as a barrier in the option value. And the optimal exercise boundary $S_f(t)$ a barrier in the asset price, which is optimal for exercise. In paper [1] the authors discussed the boundary as a moving barrier. Therefore there is a relation between the option value process and its barrier, and between the underlying asset process and its barrier. It can be expressed as follows

$$P_A(S,t) = \max(E - S, 0) \qquad if\ S < S_f(t)$$

$$P_A(S,t) \geq \max(E - S, 0) \qquad if\ S \geq S_f(t)$$

These verify that the holder of the option has to wait a period before exercising. The exercise take place when the cumulative time $(\bar{t})$ that the option value remains below the barrier reaches some pre agreed time $\bar{T}$.

Generally the barrier is crossed by the underlying asset, but in paper[1] the authors consider it as being crossed by the option value. For this new class of model, let $(\overline{dt})$ be defined as follows

$$d\bar{t} = \begin{cases} dt\ \ if\ P(S,t,\bar{t}) \leq B(S,t) \\ 0\ \ if\ \ P(S,t,\bar{t}) > B(S,t) \end{cases} \tag{2.2}$$

where $(\overline{dt})$ is the barrier clock time, it depends on whether the asset value S moves further than some fixed barrier H. B(S,t) is the barrier on the option value. The location of B(S,t) in terms of the underlying asset, which maps to some function $H(t,\bar{t})$, is given by

$$P(S,t,\bar{t}) \leq B(S,t)\ if\ S \leq H(t,\bar{t})$$

$$P(S,t,\bar{t}) > B(S,t) if\ \ S > H((t,\bar{t}) \tag{2.3}$$

Accordingly we can rewrite equation(2.2) as

$$\overline{dt} = \begin{cases} dt & if\ S \leq H(t,t) \\ 0 & if\ S > H(t,\bar{t}) \end{cases} \tag{2.4}$$

For clarity, to determine the barrier function H(t, $\bar{t}$), first we have to determine the function $P(S,t,\bar{t})$ because they depend on each other as in (2.3). That means, that the function H(t, $\bar{t}$) can be determined only, if we have obtained the value of $P(S,t,\bar{t})$. Actually this implies that the barrier H(t, $\bar{t}$) becomes a free boundary, which should be found as part of the solution. Thus the free boundary problem, is a three dimensional nonlinear problem in S,t and $\bar{t}$.

The small change in the option price, using Itô calculus can be given by

$$dP = \sigma S \frac{\partial P}{\partial S} dX + \left(\mu S \frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial^2 S} + \frac{\partial P}{\partial t}\right) dt + \frac{\partial P}{\partial \bar{t}} d\bar{t} \tag{2.5}$$

Then if we want to find the change in the option price for this new class of model, we use equation (2.2), and apply equation(2.5), we then obtain

$$dP = \begin{cases} \sigma S \dfrac{\partial P}{\partial S} dX + \left(\mu S \dfrac{\partial P}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial^2 S} + \dfrac{\partial P}{\partial t} + \dfrac{\partial P}{\partial \bar{t}}\right) dt \quad if\ S(t) < H \\ \sigma S \dfrac{\partial P}{\partial S} dX + \left(\mu S \dfrac{\partial P}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial^2 S} + \dfrac{\partial P}{\partial t}\right) dt \quad if\ S(t) > H \end{cases} \tag{2.6}$$

Assume that $dt = d\bar{t}$, and by structuring a hedging portfolio, we arrive at the following PDEs, which are applicable over different region of S and $\bar{t}$:

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial \bar{t}} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 P}{\partial S^2} + (r-d)S\frac{\partial P}{\partial S} - rP = 0, \qquad S < H \tag{2.7}$$

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-d)S\frac{\partial P}{\partial S} - rP = 0, \qquad S > H \tag{2.8}$$

We solve equation (2.7) when the asset value is below the barrier $S < H$, and (2.8) when the values of the asset is above the barrier.

Paper [1] uses the following appropriate boundary conditions for a put option in this case, together with smooth pasting condition on S=H explicitly:

$$H(t = T, \bar{t}) = E \tag{2.9}$$

$$P(S, t = T, \bar{t}) = \max[E - S, 0] \tag{2.10}$$

$$P(S, t, \bar{t} = \bar{T}) = \max[E - S, 0] \tag{2.11}$$

$$P(S = H, t, \bar{t}) = E - H \tag{2.12}$$

$$\left.\frac{\partial P}{\partial S}\right|_{S=H^+(t,\bar{t})} = \left.\frac{\partial P}{\partial S}\right|_{S=H^-(t,\bar{t})} \tag{2.13}$$

on $S = 0,\ P_t + P_{\bar{t}} - rP = 0$, as $S \to \infty$, $P(S,t,\bar{t}) \to 0$ (2.14)

The delaying of exercise is in principle a suboptimal option. This is the main reason for studying this new class of option model. The early exercise decision is not always easy for the holders.

The authors in paper [1], went also through the rational option pricing theory, where some important bounds show up, on the value of the American delayed exercise option (which placed its value between the American option and European option), this is shown as follows:

If the time-to-knockout[3]"$\bar{\tau}$", is longer than the time to expiry "τ", then the option value is equal to the European option.

$$P(S, \tau, \bar{\tau}) = P_E(S, \tau) \quad if\ \bar{\tau} \geq \tau \tag{2.15}$$

Where $\bar{\tau}$ given by $\bar{\tau} = \bar{T} - \bar{t}$, and τ is given by $\tau = T - t$

On the other hand, the value of the Ameripean option becomes equal to the value of the American option if the time to knock-out goes to zero, i.e.

$$\lim_{\bar{\tau} \to 0} P(S, \tau, \bar{\tau}) = P_A(S, \tau) \tag{2.16}$$

[3] A "time to knock-out" is the time where the option is extinguished on the price of the underlying asset breaching a barrier.

Equation (2.16) is for the extreme cases, but now, if the cumulative time ($\bar{t}$) is not of those two cases, then the option makes more chance to be exercised if the time to knock out moves closer, because in this case it must be worth more. Then the Ameripean option must satisfy

$$P(S,\tau,\bar{\tau}_1) \geq P(S,\tau,\bar{\tau}_2) \qquad \text{If } \bar{\tau}_1 \leq \bar{\tau}_2 \tag{2.17}$$

We can now write the relation between the three options using the above inequalities as follows

$$P_E(S,\tau) \leq P(S,\tau,\bar{\tau}) \leq P_A(S,\tau)$$

The new class of model explained above, takes us to an important question; what is the effect of delayed exercise on a put option? Let us go back to equation (2.3), which says; we must solve the Ameripean option as ParAsian option with moving barrier $H(t,\bar{t})$. Also to solve the PDEs (2.7) and(2.8) in the sub regions $S > H(t,\bar{t})$ and $S < H(t,\bar{t})$, respectively, we need an additional condition to be able to determine the location of the implied barrier H. But the value of the option at the barrier H, can be described using equations (2.4) and (2.12), as follows

$$P(H(t,\bar{t}),t,\bar{t}) = E - H(t,\bar{t}) \tag{2.18}$$

The second extra condition we need, is on the barrier which is given by continuity of the first derivative with respect to S when cross the barrier. That is exactly the same as the smooth-pasting conditions for the standard American option, it can be written as follows

$$\left.\frac{\partial P}{\partial S}\right|_{S=H^+(t,\bar{t})} = \left.\frac{\partial P}{\partial S}\right|_{S=H^-(t,\bar{t})} \tag{2.19}$$

So those are the conditions we need to price the option.

2.3 The perpetual case

Before starting to explain in detail the content of this subsection, let's explain the general idea of the perpetual case. As we know, the pricing formula of the American option, depends on the exercise boundary. In the perpetual case, these exercise boundaries will be described, when the maturity time tends to infinity (for more details see paper[7]).

Paper[1] has discussed the perpetual case based on the following: Consideration of the analysis for perpetual options[4] is helpful for the visions to be derived concerning the standard American and European options as limiting cases of the Ameripean option. The authors of paper [1] have used the fact that it is known that the American put option has a perpetual solution, since the value of the Put option is monotone and it has a maximum bound. Therefore, by using the rational pricing ideas from previous section, paper [1] has discussed the perpetual solution for the Ameripean option. We have seen earlier that

$$P(S,\tau_1,\bar{\tau}) > P(S,\tau_2,\bar{\tau}) \qquad \text{where } \tau_1 > \tau_2 \qquad \text{and } \tau_2 \geq \bar{\tau} \tag{2.20}$$

[4] Perpetual options: do not have an expiry date but rather an infinite time horizon. For example the Russian option and the stop-loss option. For more details see chapter 15 in book [2].

where (2.20) shows the function is monotonically increases in τ for $\tau > \bar{\tau}$, and have a upper bound

$$P(S,\tau,\bar{\tau}) < P(S,\tau) \qquad if\ \bar{\tau} > 0 \qquad (2.21)$$

The authors of paper [1] have shown, by removing the temporary time τ, i.e. taking $\frac{\partial}{\partial \tau} = 0$, in the main equations (2.7) and (2.8), then these equations fit the perpetual case (i.e. they are no longer depending on time, and become differential equations in the single variable S). So we obtain the following equations

$$P_{\bar{\tau}} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \qquad for\ S \leq H(\bar{\tau}) \qquad (2.22)$$

$$0 = \frac{1}{2}\sigma^2 S^2 \frac{\sigma^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \qquad for\ S > H(\bar{\tau}) \qquad (2.23)$$

with the following boundary conditions

$$P(S,0) = \max(E - S, 0) \qquad (2.24)$$

$$P(0,\bar{\tau}) = e^{-r\bar{\tau}}E \qquad (2.25)$$

$$P(H(\bar{\tau}),\bar{\tau}) = E - H(\bar{\tau}) \qquad (2.26)$$

$$\left.\frac{\partial P}{\partial S}\right|_{S=H^-(\bar{\tau})} = \left.\frac{\partial P}{\partial S}\right|_{S=H^+(\bar{\tau})} \qquad (2.27)$$

$$P(\infty,\bar{\tau}) = 0 \qquad (2.28)$$

The authors of paper [1] have examined equation (2.23), which can be analytically solved using a solution of the Euler type. The solution of Euler type, can be written as follows:

$$P(S,\bar{\tau}) = (E - H)(\frac{S}{H})^\alpha \qquad (2.29)$$

where α should be equal to $\alpha = -\frac{2r}{\sigma^2}$, so that the solution (2.29) can satisfy the zero boundary condition (2.28) at infinity. This solution, for a given value $H(\bar{\tau})$, will have a value for the first derivative, which will fit in equation (2.22) using the boundary condition (2.27). Hence, equation (2.22), must be solved numerically in the region $S \leq H(\bar{\tau})$ using the following boundary conditions

$$P(S,0) = \max(E - S, 0) \qquad (2.30)$$

$$P(0,\bar{\tau}) = e^{-r\bar{\tau}}E \qquad (2.31)$$

$$P(H,\bar{\tau}) = E - H \qquad (2.32)$$

$$\left.\frac{\partial P}{\partial S}\right|_{S=H} = \alpha(E - H) \qquad (2.33)$$

where equation (2.33) arises from equation (2.29).

Moreover, an extra condition for the initial value of the free boundary is needed. Also paper [1] has shown that the free barrier H goes to the free boundary S_f when $\bar{\tau} \to 0$, and then

$$H(0) = E\frac{\alpha}{\alpha-1} \tag{2.34}$$

An asymptotic analysis, in the limit as time-to-knockout tends to zero, produces an ordinary differential equation (ODE). This ODE matches both of the boundary conditions at the free boundary (see Appendix C in paper [1]) and allows the authors of [1] express the free boundary as follows

$$H(\bar{\tau}) = E\frac{\alpha}{\alpha-1}\left(1+\varepsilon_0\sqrt{\frac{1}{2}\sigma^2\bar{\tau}}+\cdots\right) \qquad as\ \bar{\tau} \to 0 \tag{2.35}$$

where $\bar{\tau}$ is the time-to-knockout and ε_0=0.9034466.

3. Numerical Method

The option valuation can be described by the PDEs (2.7) and (2.8), linked by the continuity of the option value and its delta hedging, as has been given in paper [1].

Equation (2.8) can be solved similarly as an American Put option using a special method called "Enhanced Boundary Fitted Coordinates (EBFC)" scheme of Johnson in [7], this scheme is based on the "Boundary Fitted Coordinates(BFC)" scheme. This method is needed since the numerical analysis of the Ameripean delayed-exercise option can be complicated. A description of the technique is given in two subsections, first using BFC to solve the standard American option for simplicity, and then using EBFC to solve the Ameripean option.

3.1 The Boundary Fitted Coordinates "BFC"

The BFC method applies the transformation ($\tau = T - t$) idea to solve the moving boundary problem over a fixed coordinate system. Crank-Nicolson were the first to apply this idea to a finite-difference scheme to solve the diffusion equations, see [2] and [7] for more details. Widdicks (2002) has modified the standard finite-difference scheme to solve the American option problem. The advantage of BFC method is that, at each time step it solves accurately and explicitly the position of the free boundary. This is important because the holder of the option needs to know when it is optimal to exercise exactly. On the other hand this method has difficulty to follow the free boundary when the time to expiry is small. The same holds for many other numerical methods. Because of this reason, paper [7] suggested a new method that combines the BFC method with a "curtailed range" analysis, similar to method developed in paper [9] for lattice methods, to have a method which is both accurate and does not have problems with the length of the time steps.

Paper [1] introduced the following transformation for S for the American put option:

$$\hat{S} = S - S_f(\tau) \tag{3.1}$$

Where $\hat{S} \in [0, \infty]$ and if $S = S_f(t)$ then $\hat{S} = 0$.

By using this transformation, equation (2.8) becomes:

$$\frac{\partial P}{\partial \tau} - \frac{\partial S_f}{\partial \tau}\frac{\partial P}{\partial \hat{S}} = \frac{1}{2}\sigma^2(\hat{S} + S_f)^2\frac{\partial^2 P}{\partial \hat{S}^2} + r(\hat{S} + S_f)\frac{\partial P}{\partial \hat{S}} - rP \tag{3.2}$$

Clearly, equation (3.2) is nonlinear, because of the unknown function $S_f(t)$ which actually has to be found as a part of the solution. Paper [7] has used Newton iteration to linearize this equation.

3.2 The enhanced boundary fitted coordinate (EBFC)

Since the BFC method has difficulties to overcome with the free boundary when the time to expiry is small, another method has been used. The author of [10] suggests if the two boundaries converge to the same point, then the difficulties appear with tracking the boundary near expiry will be solved, and the BFC method can follow accurately the boundaries also in these limits. If there is some possibility to define an upper bound on the solution for a put, the problem can be formulated with one known boundary and one free boundary. If the two boundaries converge to the same point, then all difficulties appearing with tracking the boundary near expiry can be solved. Although the nodes[5] in the transformed space have equal distances, the distance between nodes in real space will become small near expiry.

As we said earlier the “curtailed range” technique can be applied to “lattice” and “finite-difference grid” methods. Paper [1] has used the curtailed technique with some adaptation. The adaptation is that, instead of fitting the coordinate system around the curtailed range they have discounted nodes over which the solution is obtained. The suggested curtailed range is

$$S_{max} = \max\left[S_0 e^{r\tau+\varepsilon\sigma\sqrt{\tau}}, E e^{-r\tau+\varepsilon\sigma\sqrt{\tau}}\right] \tag{3.3}$$

$$S_{min} = \min\left[S_0 e^{r\tau-\varepsilon\sigma\sqrt{\tau}}, E e^{-r\tau-\varepsilon\sigma\sqrt{\tau}}, 0\right] \tag{3.4}$$

where ε is a parameter defined by the user and has an effect on the accuracy of the result. This parameter measures the number of the standard deviations in the normal distribution it will be from the mean. Such a region for a put option defined by $S \in [S_{max}, \infty]$, thus increasing ε will make the probability of the option being exercised at S_{max} very small, and then the accuracy will increase. Therefore paper [1] has denoted by $S_{max} = U(\tau)$ the upper boundary, and specified it for the American put option by

$$U(\tau) = \min\left[E e^{\varepsilon\sigma\sqrt{\tau}}, \lambda E\right] \tag{3.5}$$

where λ is a constant defined by the maximum value of the grid, we can see that the term $-r\tau$ is cancelled from equation (3.3) so that $U(\tau) > E$. Now denote by $F(\tau) = L(\tau)$ the lower boundary, and introduce

$$D(\tau) = U(\tau) - L(\tau) \tag{3.6}$$

to be the difference between the boundaries. Also introduce the mapping from $[S_{min}, S_{max}] \rightarrow [0,1]$ by

$$\hat{S} = \frac{S - L(\tau)}{D(\tau)}. \tag{3.7}$$

The authors of paper [1] have specified L and D according to the option under consideration. Now for the American put option , the upper bound has been defined earlier in equation (3.5), while the lower boundary $F(\tau) = S_f(\tau)$ is the free boundary, hence equation (3.6) becomes

$$D(\tau) = U(\tau) - S_f(\tau)$$

So it is now clear why this transformation method will be useful and accurate: as paper [1] explains, other methods are facing problems when time tends to expiry because ΔS cannot be chosen small

[5] a node describes the position of the asset price at a particular time.

enough to capture the solution in this limit, however the EBFC method gives the solutions with a high level of accuracy as expiry is approached (=$\tau \to 0$), and that is because the grid spacing is small.

So for the American put option, when $S > H(\tau, \bar{\tau})$ then the problem describes simply the standard American put option, so we can use the method just described for that option (i.e. BFC method). In paper [1] the authors have chosen U to be in this case in the form

$$U = Ee^{\varepsilon\sigma\sqrt{\tau}} \tag{3.8}$$

Now by using the transformation in equation (3.7), and equation (3.6) on the Black-Scholes equation (2.8) we get

$$V_\tau - \frac{1}{D}\left(\hat{S}D_\tau + L_\tau\right)V_{\hat{S}} = \frac{1}{2}\sigma^2(\hat{S} + \frac{L}{D})^2 V_{\hat{S}\hat{S}} + r\left(\hat{S} + \frac{L}{D}\right)V_{\hat{S}} - rV \tag{3.9}$$

where V defines the solution above the barrier, and $V_\tau = \frac{\partial V}{\partial \tau}$, $V_{\hat{S}} = \frac{\partial V}{\partial \hat{S}}$ and $V_{\hat{S}\hat{S}} = \frac{\partial^2 V}{\partial \hat{S}^2}$.

Now if $S < H(\tau, \bar{\tau})$, then the boundary conditions correspond to the conditions of the American call option (For more details see paper [1]).

The option values for the put option above the barrier are solved using the following boundary conditions:

$$F(0, \bar{\tau}) = E \tag{3.10}$$

$$V(\hat{S}, \tau, 0) = \max\left[E - \left(\hat{S}D + F\right), 0\right] \tag{3.11}$$

$$V(0, \tau, \bar{\tau}) = E - F(\bar{\tau}) \tag{3.12}$$

$$\frac{\partial V}{\partial \hat{S}}(1, \tau) = -D(\tau) \tag{3.13}$$

$$V\left(\hat{S} = 1, \tau, \bar{\tau}\right) = 0 \tag{3.14}$$

$$D(\tau, \bar{\tau}) + F((\tau, \bar{\tau}) = U(\tau) \tag{3.15}$$

Simply , if we suppose to have n nodes above the barrier and m below, then in total we get n+m+2 equations. The extra 2 equation are to obtain the F and D, also to ensure that the solutions match at the boundary. So both problems (above/below barrier), can be discretized using the finite-differencing coupled to Crank-Nicolson method. But this generates a set of nonlinear equations, which can be solved using the Newton's method, where the solution at τ, $\bar{\tau}$ step gives effective first guess in the iteration process. Paper [1] said, in order to preserve second-order converge, then $\Delta\tau = \Delta\bar{\tau}$, where the grid point exists along the characteristic line $\tau = \bar{\tau} + constat$.

4. Results

All of the following results are for a delayed-exercise Put option (Ameripean put option) as standard case. Paper [1] has obtained the results using the following values, for the interst rate r=0.4, for the strike price E=1, for the expiry time T=0.5, and for the volatility of the underlying asset σ=0.4. The figures below (figures 2-4 in paper [1]) show that the delayed exercise option meets all the criteria set down in the rational option pricing theory. The figures show that, if the time-to-knockout increases, then the value of the option decreases (see fig 2). This can be seen as follows: if the option is less likely to be exercised then the option is worth less.
The figure also shows that the free-boundary barrier (both H and S_f) is continuous, and when the time-to-knockout decreases it tends to the American free-boundary, and the value of the option is continuous as well.

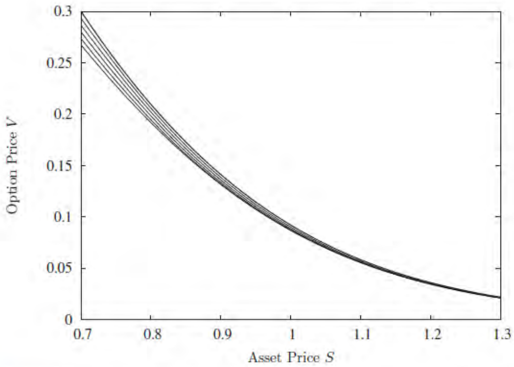


Figure 2. *The Ameripean put option. A fluid link from the corresponding American to the European option. From top to bottom T = 0 (American), 0.1, 0.2, 0.3, 0.4, and 0.5 (European). Parameters are σ = 0.4, r = 0.1, T = 0.5, and E = 1.*

The following figure (fig 3 in paper[1]), shows the exercise premium $\varepsilon(S, t, \bar{t})$, which is computed by calculating the difference between the Ameripean option and the equivalent European option as paper [1] gives.

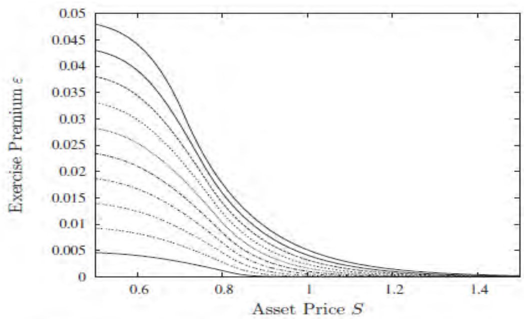


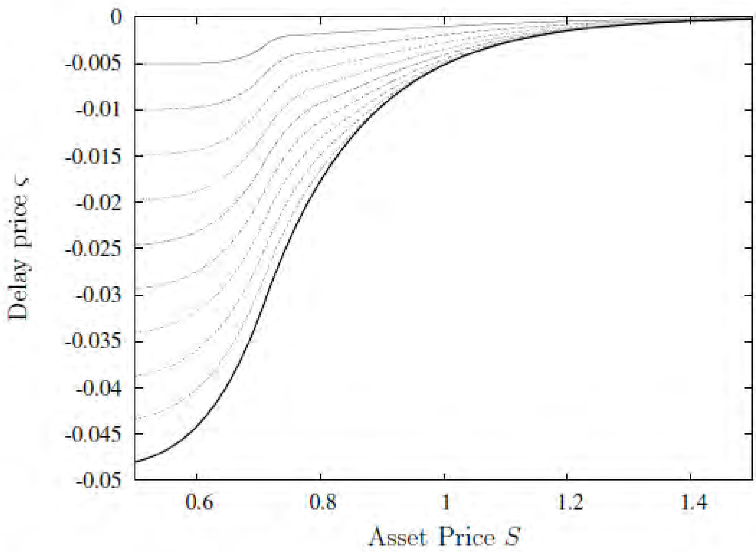
Figure 3. *The exercise premium. The value of the option to exercise $(V(S,0,0) - V_E(S,0))$ for various knockout times T against asset price S. The thick solid line is the value of the American option to exercise $(V_A - V_E)$. From top to bottom the knockout time T is increased in steps of 0.05, from 0 (American) to 0.5 (European). Parameters are σ = 0.4, r = 0.1, T = 0.5, and E = 1.*

As we can see, when S is zero, the exercise premium is computed by

$$\varepsilon(0, t, \bar{t}) = \left(\min\left[e^{-r(\bar{\tau})}, e^{-r(\tau)}\right]\right) E$$

when S=0, then the asset price remains under the implied asset barrier, which says that, the exercise can take place either at expiry or after the time-to –knockout $\bar{\tau}$ when $\bar{\tau} < \tau$. When the probability that the asset price will remain under the implied barrier for the time-to-knockout $\bar{\tau}$ tends to zero, then, when the asset value price grows, the value of the option to exercise is decreased. And hence, the Ameripean and the American option values become similar to the European option value when the exercise premium goes to zero(i.e. $\varepsilon \rightarrow 0$).

Figure 4 shows the delay option exercise line, as paper [7] explains, it has been calculated, by taking the difference between the delayed- exercise option and the American option. From the figure we can see that, when the time-to-knockout is small, then the time that exercise has to delay is also small, that means the option to delay value is small.



Figuer 4

The delay option.
The value of the option to delay $(V(S,0,0) - V_A(S,t))$ against the asset price S. The thick solid line is the value of the option to wait till expiry $(V_E - V_A)$. Top to bottom time-to-knockout $\bar{T}$ is increased in steps of 0.05 from 0 (American) to 0.5 (European).

In this region it is clear that, when the asset value is under the barrier, the probability of exercising the option is increased, and vice versa, even when it is just above the barrier.

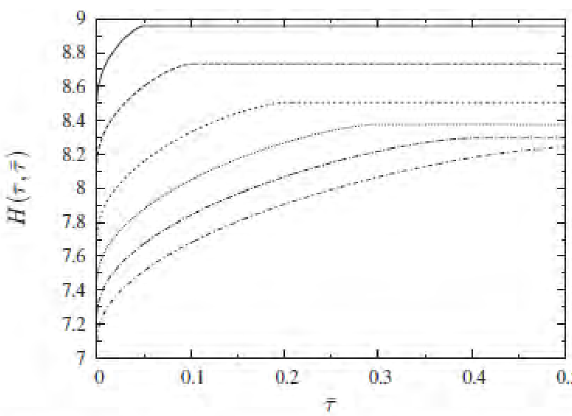
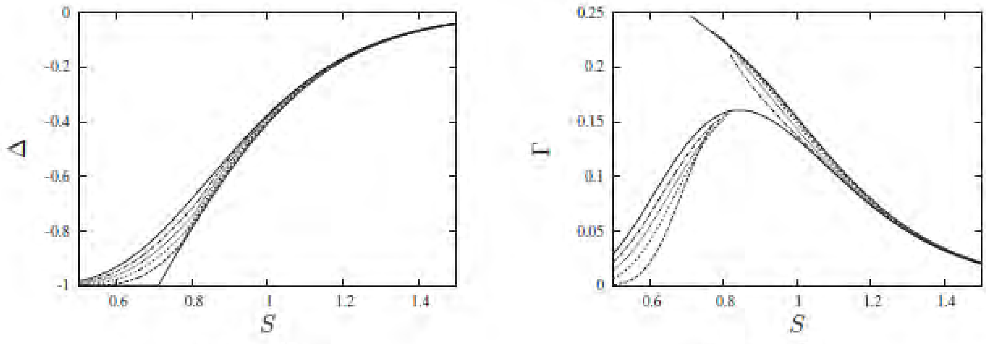


Figure 5 *The free barrier. The free "barrier" for the Ameripean option versus time-to-knockout, for varying time-to-expiry. Top to bottom,* $\tau = 0.05,\ 0.1,\ 0.2,\ 0.3,\ 0.4,$ *and* 0.5. *Parameters are* $\sigma = 0.4$, $r = 0.1$, $T = 0.5$, *and* $E = 1$.

Figure 5 explain the free barrier location $H(\tau, \bar{\tau})$ using different values for the time-to –expiry (τ) for varying time-to-knockout ($\bar{\tau}$). When $\bar{\tau} \geq \tau$ then we get a flat curve, since $H(\tau, \bar{\tau}) = H_E(\tau)$, i.e. the intersection between the European value and payoff is not possible. In this region when $\bar{\tau} \to 0$, the option is more likely to become American option with
$H(\tau, \bar{\tau}) \to S_f(\tau)$.



(a) $\Delta = \frac{\partial V}{\partial S}$.

(b) $\Gamma = \frac{\partial^2 V}{\partial S^2}$.

Figure 6 . *The greeks. The value of the* Δ *and* Γ *against the asset price* S *for varying values of time-to-knockout for a six-month option. Top to bottom,* $T = 0.5$ *(European), 0.4, 0.3, 0.2, 0.1, and 0 (American). Parameters are* $\sigma = 0.4$, $r = 0.1$, $T = 0.5$, *and* $E = 1$.

Figure 6 gives the Greeks graphs, it shows in 6(a) delta, where the delta for the American put option no longer satisfies the Black-Scholes equation in the region where $S < S_f(t)$ since the option has already exercised, for more details see book [2]. This gives clearly a discontinuity in the gamma see figure 6(b), this discontinuity is moved to the American Γ at the barrier $H(\tau, \bar{\tau})$. The delta shows continuity up to the barrier, also it follows the hedging strategy for the American option in the region $S > H$, and it looks like a European option when $S < H$. This kind of strategy changes across the barrier, this is the main reason for the discontinuity in the gamma.

5. Computation of the free boundary of an American put option

In the previous sections we have presented the definition, the numerical solution and the results for the “Ameripean Delayed-exercise” model. As explained in sections three and four, finding the delta is important because it gives the rate of change of the option value with respect to the underlying asset, and finding the free-boundary $S_f(t)$ is also important because it gives the optimal exercise price.

In this section we are going to focus on computing the optimal exercise boundary (free- boundary ($S_f(t)$) particularly for the American Put option.

As we know the American option gives its holder the right to exercise the option during the whole period till the expiry date. Therefore, the valuation of the American option is more complicated than the European option. At each time step we have to compute the option value, and for each value S we have to see whether the option should be exercised or not. That means, at each time t there is a value for S, which represent the boundary between two regions, where in one region the holder should exercise the option, and in the other region he has to hold the option. Book [2] has denoted the value of S where the holder can exercise by $S_f(t)$, and refers to it as the optimal exercise price.

Since it is difficult to find a solution to a free boundary problem, the authors of book [2] have defined a theoretical framework within which to explain the free boundary problem in general terms. Two methods have been presented for the numerical solution. The first one is to track the free boundary as a part of the time stepping process. The second one is finding a transformation that reduces the free boundary problem to a fixed boundary problem from which we can get the free boundary after that. Book [2] explained that the first method is not attractive for the America option, because, it does not give expression for the free boundary condition or its time derivative. Chapter 7 and 9 in [2] explained also how to apply the second method. There are many transformations that can do this, but we will particularly used the method involving the use of the linear complementarity formulation.

5.1 Linear complementarily problem for the American option

To simplify the numerical solution method, first we have to transform the American put option variables as follows: form the original (S,t) variables to (x, τ), where

$S = Ee^x$

and (5.1)

$t = T - \tau / \frac{1}{2}\sigma^2$ (5.2)

also the function $S = S_f$ becomes $x = x_f$

then the payoff function max[E-S,0] becomes

$$g(x,\tau):=e^{\frac{1}{2}(k+1)^2\tau}\max(e^{\frac{1}{2}(k-1)x}-e^{\frac{1}{2}(k+1)x},0) \quad (5.3)$$

Now the problem is how to find a solution for

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2} \qquad for\ x > x_f(\tau) \quad (5.4)$$

$$u(x,\tau)=g(x,\tau) \qquad for\ x \leq x_f(\tau) \quad (5.5)$$

using the following initial condition

$$u(x,0)=g(x,0)=\max(e^{\frac{1}{2}(k-1)x}-e^{\frac{1}{2}(k+1)x},0) \quad (5.6)$$

and the behaviour

$$\lim_{x\to\infty} u(x,\tau)=0 \quad (5.7)$$

for $x \to -\infty$, this means early exercise is optimal, and so $u = g$. In addition we have the constraint

$$u(x,\tau) \geq e^{\frac{1}{2}(k+1)^2\tau} max(e^{\frac{1}{2}(k-1)x}-e^{\frac{1}{2}(k+1)x},0) \quad (5.8)$$

and the conditions that $\frac{\partial u}{\partial x}$ be continues at $x = x_f(t)$.

For a numerical reasons [2] has suggested that the equations (5.3)-(5.8) only for x in the interval $-x^- < x < x^+$.

We can write the equations (5.3)-(5.8) in the linear complementarity form as follows

$$\left(\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}\right).\big(u(x,\tau)-g(x,\tau)\big)=0 \quad (5.9)$$

this means that

$\left(\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}\right) \geq 0$ or $\big(u(x,\tau)-g(x,\tau)\big) \geq 0$

With initial conditions (5,6) and

$$u(x^+,\tau)=0, \quad u(-x^-,\tau)=g(-x^-,\tau) \qquad for\ all\ \tau \quad (5.10)$$

and the continuity condition of u and $\frac{\partial u}{\partial x}$. This translates into the assumption, that for large value for S we have P=0, and for small value P=E-S. So when (u=g) its optimal to exercise and when (u>g) it is not.

5.2 Numerical solution of the problem

We are going to solve the problem by setting up a grid and using finite-difference method to the partial derivatives. If we can solve the linear complementarity problem then we find $x_f(\tau)$ by using the condition

$$u\left(x_f(\tau),\tau\right) = g\left(x_f(\tau),\tau\right) \qquad (5.11)$$

but

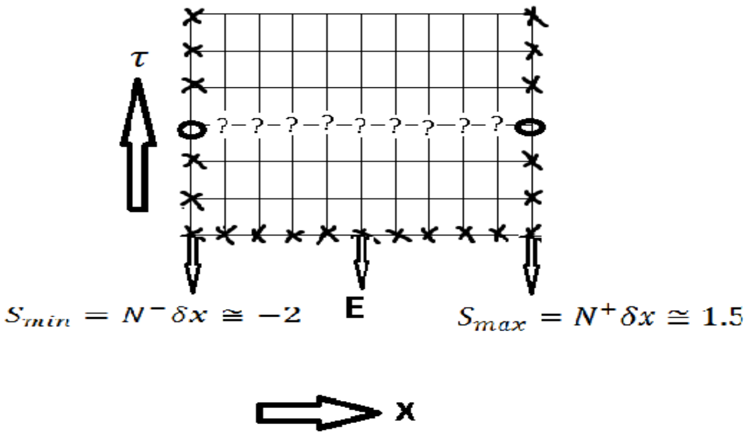
$$u(x,\tau) > g(x,\tau) \qquad voor\ x > x_f(\tau) \qquad (5.12)$$

Then the free boundary is defined by the points where $u(x,\tau)$ first meets $g(x,\tau)$.

5.2.1 Finite difference formulation

Using the regular finite mesh, and truncate x so that x is between $N^-\delta x$ and $N^+\delta x$,

$$N^-\delta x \leq x = n\delta x \leq N^+\delta x \qquad (5.13)$$



where δx and $\delta\tau$ are the step sizes, and by using Crank-Nicolson as numerical method[6] on approach of linear complementarity method as it is explained in chapter 7,8, and 9 in book [2], let

$$u_n^m := u(n\delta x, m\delta\tau) \qquad (5.14)$$

then write

$$\frac{\partial u}{\partial \tau}\left(x,\tau+\frac{\delta\tau}{2}\right) = \frac{u_n^{m+1}-u_n^m}{\delta\tau} + O(\delta\tau)^2) \qquad (5.15)$$

and

$$\frac{\partial^2 u}{\partial x^2}\left(x,\tau+\frac{\delta\tau}{2}\right) = \frac{1}{2}\left(\frac{u_{n+1}^{m+1}-2u_n^{m+1}+u_{n-1}^{m+1}}{(\delta x)^2}\right) + \frac{1}{2}\left(\frac{u_{n+1}^{m}-2u_n^{m}+u_{n-1}^{m}}{(\delta x)^2}\right) + O((\delta x)^2) \qquad (5.16)$$

If we neglect the terms $O(\delta\tau)^2)$ and $O((\delta x)^2)$, then the inequality $\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0$ can be approximated by

$$u_n^{m+1} - \frac{1}{2}\alpha(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}) \geq u_n^m + \frac{1}{2}\alpha(u_{n+1}^m - 2u_n^m + u_{n-1}^m) \qquad (5.17)$$

[6] we can use any other numerical method(e.g. LU or SOR), we have only to be careful in choosing the size of the steps, it should be small enough to get a good result.

where $\alpha = \frac{\delta\tau}{(\delta x)^2}$ (5.18)

We can also write $g_n^m = g(n\delta x, m\delta\tau)$ for the discretized payoff function. Then

$$u(x,\tau) \geq g(x,\tau) \quad (5.19)$$

becomes

$$u_n^m \geq g_n^m \qquad for\ m \geq 1 \quad (5.20)$$

The boundary and the initial conditions imply the following

$$u_{N^-}^m = g_{N^-}^m, \qquad u_{N^+}^m = u_{N^+}^m, \qquad u_n^0 = g_n^0 \quad (5.21)$$

Let us defined Z_n^m by

$$Z_n^m = (1-\alpha)u_n^m + \frac{1}{2}\alpha(u_{n+1}^m + u_{n-1}^m)$$

Then we can rewrite (5.17) as follows

$$(1+\alpha)u_n^{m+1} - \frac{1}{2}\alpha(u_{n+1}^{m+1} + u_{n-1}^{m+1}) \geq Z_n^m$$

such that at each time step $(m+1)\delta\tau$ we can simply find Z_n^m since we know u_n^m. Condition (5.9) can be approximated by

$$\left((1+\alpha)u_n^{m+1} - \frac{1}{2}\alpha(u_{n+1}^{m+1} + u_{n-1}^{m+1}) - Z_n^m\right)(u_n^{m+1} - g_n^{m+1}) = 0$$

Matlab code

we have computed $x_f(\tau)$ using matlab programme. The main idea was, to compute the point $u(n\delta x,(m+1)\delta\tau)$ given the point $u(n\delta x, m\delta\tau)$, in other word is to check whether

$$u(n\delta x, m\delta\tau) = g(n\delta x, m\delta\tau)$$

or

$$u(n\delta x, m\delta\tau) > g(n\delta x, m\delta\tau)$$

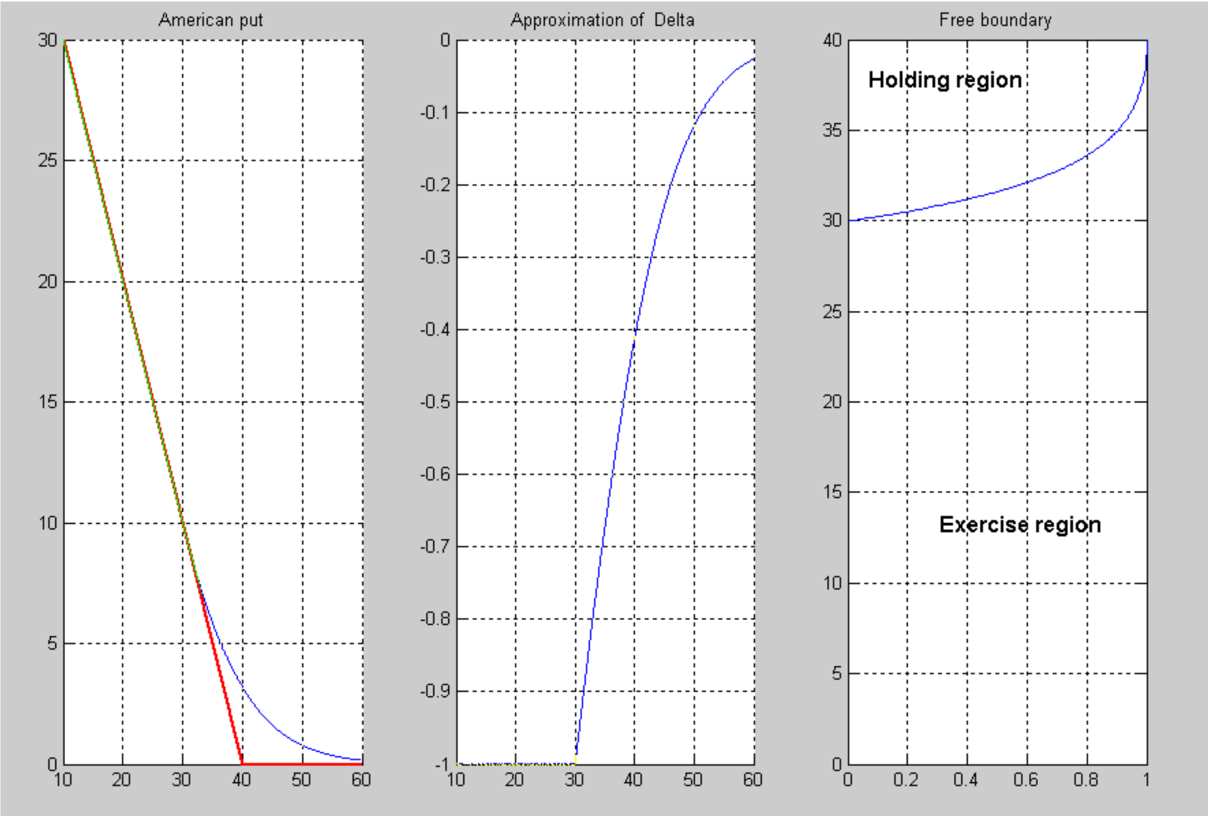
Then $x_f(m\delta\tau)$ is equal to the maximum point where $u(n\delta x, m\delta\tau) = g(n\delta x, m\delta\tau)$. After that, we return the transformation back to $S_f(\tau) = Ee^{x_f(\tau)}$, i.e. by increasing x we increase S, and as soon as P(S,t)>E-S we have detected the value of $S_f(t)$.

The code:

```
z=10^(-8);
Q=u1;
```

```
 j=1;
%to find the optimal value for exersing the option, and the free        %boundary values
while abs(Q(j)-gLCP(j))<z
      j=j+1;
  end

  Sf(i)=Exercise*exp(xarg(j))
```



6. Conclusion and Summary

The Ameripean option model proposed in this paper is interesting as a model providing a link between the American and European options. New numerical methods are applied to solve an option which is highly path dependent. These methods are efficient at tracking the free boundaries.

In particular the free boundary for the American put option is explained and results are provided to find it. This is done using the finite difference method, and taking into account the size of the steps, which should be small enough to get an accurate result.

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